Heteroclinic Cycles for Reaction Diffusion Systems by Forced Symmetry-Breaking

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Abstract. We consider solutions of the semilinear parabolic equation (1.1) on the 2-Sphere. Assuming (1.1) has an axisymmetric equilibrium u_α, the group orbit of u_α gives a whole (invariant) manifold M of equilibria for (1.1). Under generic conditions we have that, after perturbing (1.1) by a (small) $L \subset \mathbf{O}(3)-$equivariant perturbation, M persists as an invariant manifold $\widetilde{M}$ slightly changed. However, the flow on $\widetilde{M}$ is in general no longer trivial. Indeed, we find heteroclinic orbits on $\widetilde{M}$ and, in case $L = \mathbb{T}$ (the tetrahedral subgroup of $\mathbf{O}(3)$), even heteroclinic cycles.

1 Introduction: A Motivating Example

Recently, $L-$equivariant flows on homogeneous spaces G/H, where G is a compact Lie group and L, H are subgroups, have been of high interest, since it seemed possible to derive by this group theoretical approach information on heteroclinic orbits, even in PDE's. A seminal presentation of these ideas in the case $G = \mathbf{SO}(3)$ can be found in Lauterbach and Roberts [12].

In order to motivate our group theoretical discussions of the following sections, we consider solutions $u = u(t,x), x \in S^2 \subset \mathbb{R}^3, t \geq 0$ of the semilinear parabolic equation on the 2-sphere

$$u_t = A(\lambda)u + f(u) \quad =: g(u,\lambda). \tag{1.1}$$

Here $f : \mathbb{R} \to \mathbb{R}$ is a smooth nonlinearity with $f(0) = 0$ and $f'(0) = 0$. $A(\lambda) : D \subset L^2(S^2) \to L^2(S^2)$ is a linear, symmetric operator (depending continuously on a parameter $\lambda \in \mathbb{R}$) with $-A(\lambda)$ sectorial. Thus $A(\lambda)$ generates an analytic semigroup (cf. [6], Chapter 3). Moreover, we assume that $A(\lambda)$ is $\mathbf{O}(3)-$equivariant, and therefore

$$g(\gamma u, \lambda) = \gamma g(u,\lambda) \quad \text{for all} \quad \gamma \in \mathbf{O}(3), \tag{1.2}$$

where the standard action $\gamma u(x) := u(\gamma^{-1}x)$ of $\mathbf{O}(3)$ on $L^2(S^2)$ is used. So one may think of $A(\lambda) = \Delta - \lambda \mathrm{Id} : H^2(S^2) \to L^2(S^2)$, where Δ is the Laplace-Beltrami operator, but also equations like Cahn-Hilliard equations (cf. [14]) on the 2-Sphere fit into our concept.

Equation (1.1) generates a $G = \mathbf{O}(3)-$equivariant semi–dynamical system

$$\Phi^\lambda : \mathbb{R}^+ \times L^2(S^2) \to L^2(S^2). \tag{1.3}$$

Obviously, $f(0) = 0$ implies that we have the trivial solution $u \equiv 0$ for all $\lambda \in \mathbb{R}$ in (1.1), since $g(0,\lambda) = 0$. If we assume that $A(\lambda_0)$ has a nontrivial kernel, we obtain under additional conditions (e.g. a transversality condition cf. [5], Theorem 3.5; an existence result in case the domain of Equation (1.1) is a ball instead of the sphere S^2 can be found in [10]) that the equation

$$g(u, \lambda) = 0 \tag{1.4}$$

has a branch of nontrivial solutions $(u_\alpha, \lambda_\alpha)$ near $(0, \lambda_0)$ (for α in a neighborhood of zero) which all have the same isotropy subgroup $H = \Sigma_{u_\alpha} = \{\gamma \in G | \gamma u_\alpha = u_\alpha\}$. Without loss of generality, we write

$$u_\alpha = \alpha u^* + o(\alpha) \text{ for } \alpha \text{ near } 0, \tag{1.5}$$

with $u^* \in \ker A(\lambda_0)$ and $\Sigma_{u^*} = H$. The group orbit of u_{α_0} for α_0 fixed

$$\mathcal{O}(u_{\alpha_0}) := \{\gamma u_{\alpha_0} | \gamma \in G\} \cong G/H \tag{1.6}$$

gives a whole branch of solutions of (1.4), and therefore of equilibria of (1.1). Since the flow $\Phi^{\lambda_{\alpha_0}}$ of (1.3) on $\mathcal{O}(u_{\alpha_0})$ is trivial, $\mathcal{O}(u_{\alpha_0})$ is an invariant set for $\Phi^{\lambda_{\alpha_0}}$, and the semi–dynamical system (1.3) may be restricted to $\mathcal{O}(u_{\alpha_0})$. Due to the compactness of $\mathcal{O}(u_{\alpha_0})$ it gives a dynamical system

$$\Phi^{\lambda_{\alpha_0}} : \mathbb{R} \times \mathcal{O}(u_{\alpha_0}) \to \mathcal{O}(u_{\alpha_0}). \tag{1.7}$$

This simple and rather boring situation changes dramatically, once we add a (small) symmetry-breaking term in (1.1). Consider

$$u_t = A(\lambda)u + f(u) + \varepsilon h(u) \quad =: g_\varepsilon(u, \lambda), \tag{1.8}$$

where $\varepsilon > 0$ is a small parameter and $h : D \subset L^2(S^2) \to L^2(S^2)$ is a smooth L–equivariant mapping. In the case that $\mathcal{O}(u_{\alpha_0})$ is a normally hyperbolic manifold with respect to the flow $\Phi^{\lambda_{\alpha_0}}$, this invariant manifold persists, slightly changed, for the perturbed equation (1.8) with $\varepsilon > 0$ sufficiently small (cf. Proposition 1.1 in [12] and [7] for the concept of a normally hyperbolic manifold). That means there exists a manifold $M^{\varepsilon,\alpha_0} \subset L^2(S^2)$, which is L–equivariantly diffeomorphic to $\mathcal{O}(u_{\alpha_0})$ and therefore to G/H, such that the perturbed L–equivariant flow $\tilde{\Phi}^{\varepsilon,\lambda_{\alpha_0}}$, generated by (1.8) with $(\varepsilon, \lambda_{\alpha_0})$, is invariant on M^{ε,α_0}:

$$\tilde{\Phi}^{\varepsilon,\lambda_{\alpha_0}} : \mathbb{R} \times M^{\varepsilon,\alpha_0} \to M^{\varepsilon,\alpha_0}. \tag{1.9}$$

The hypotheses to guarantee that the manifold is normally hyperbolic will generically be satisfied (cf. [3], Theorem A.20). Although the unperturbed flow $\Phi^{\lambda_{\alpha_0}}$ was trivial on

$\mathcal{O}(u_{\alpha_0})$, this is in general no longer the case for $\tilde{\Phi}^{\varepsilon,\lambda_{\alpha_0}}$ on M^{ε,α_0}. For that reason we will study $L-$equivariant flows Ψ on G/H

$$\Psi : \mathbb{R} \times G/H \to G/H, \tag{1.10}$$

with L and H subgroups of a compact Lie group G (cf. e.g. [11] for more information on that topic). $L-$equivariance is still a very severe restriction, since subsets of G/H, which are fixed under subgroups L' of L

$$Fix_{G/H}(L') := \{y \in G/H \mid ly = y \ \forall l \in L'\} \subset G/H \tag{1.11}$$

are necessarily invariant under the flow Ψ (cf. Proposition 1.6 in [12]). For instance, if $G = \mathbf{SO}(3)$ and $H = \mathbf{O}(2)$ we obtain $G/H \cong \mathbb{P}^2$, the two dimensional real projective space. $L = \mathbb{T}-$equivariant flows on $\mathbb{P}^2$ are shown in the following figure (cf. [12], Section 2.2):

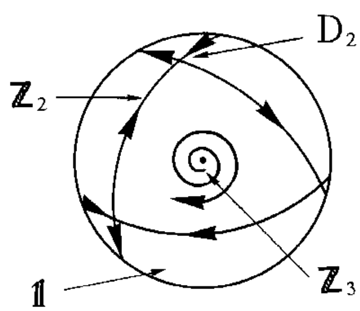

Figure 1: Fig. 3 from Lauterbach and Roberts
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Here the nontrivial subgroups of $\mathbb{T}$ are three copies of $L' = \mathbb{Z}_2$, four copies of $L' = \mathbb{Z}_3$ and one copy of $L' = D_2$. The last one is the disjoint union of all $\mathbb{Z}_2$ subgroups. It turns out that $Fix_{\mathbf{SO}(3)/\mathbf{O}(2)}(\mathbb{Z}_2) \cong S^1 \dot{\cup} 1pt$, $Fix_{\mathbf{SO}(3)/\mathbf{O}(2)}(\mathbb{Z}_3) \cong 1pt$ and $Fix_{\mathbf{SO}(3)/\mathbf{O}(2)}(D_2) \cong 3pt$ (we use 'pt' as abbreviation for isolated points). As it is indicated, the isolated points in $Fix_{\mathbf{SO}(3)/\mathbf{O}(2)}(\mathbb{Z}_2)$ are fixed by D_2.

The isolated points in $Fix_{G/H}(L')$ (for some subgroup L' in L) play a special role: By continuity of the flow, these isolated sets also have to be invariant sets for every $L-$equivariant flow. That means all these points give equilibria for $L-$equivariant flows. We call these points *equilibria of* $(L, G/H)$ and write:

$$\mathcal{E}_{(L,G/H)} := \{y \in G/H \mid y \text{ is isolated in its stratum } \},$$

i.e. for $y \in \mathcal{E}_{(L,G/H)}$ exists some subgroup $L' \subset L$ such that y is an isolated component of $Fix_{G/H}(L')$. Also of great interest are the points connecting two such equilibria of the group. We call a set $\Upsilon \subset Fix_{G/H}(L') \subset G/H$ (for some subgroup $L' \subset L$) a *connection of* $(L, G/H)$, if $Fix_{G/H}(L')$ contains some isolated subset diffeomorphic to S^1 and Υ has the form

$$\Upsilon = \{\omega(\varphi) \mid \varphi \in (0, \varphi^*)\} \subset S^1 \subset Fix_{G/H}(L'), \tag{1.12}$$

where $\omega : [0, \varphi^*] \to S^1$ is an injective smooth mapping with $\omega(0), \omega(\varphi^*) \in \mathcal{E}_{(L,G/H)}$ but $\omega(\varphi) \notin \mathcal{E}_{(L,G/H)}$ for all $\varphi \in (0, \varphi^*)$. Let

$$\mathcal{H}_{(L,G/H)} := \{\Upsilon \mid \Upsilon \text{ is a connection of } (L, G/H)\}. \tag{1.13}$$

Of course connections Υ of the group need not be heteroclinic orbits of an $L-$equivariant flow, but since Υ is an invariant set for all these flows, there is a good change to find a flow having a heteroclinic orbit on Υ.

In Figure 1 the equilibria of $(\mathbb{T}, \mathbf{SO}(3)/\mathbf{O}(2))$ are shown in bold face and the connections of $(\mathbb{T}, \mathbf{SO}(3)/\mathbf{O}(2))$ connect them.

The aim of this paper is to prove results about the flow on these connections of $(L, G/H)$. It will turn out that, indeed there is a restriction for the flow on these parts, if the symmetry-breaking in (1.8) is sufficiently small. To that end, in Section 2 we derive a formula which enables us to calculate flows on connections of $(L, G/H)$ for small symmetry-breaking. The rest of the paper is dedicated to applications of this flow formula in the case $G = \mathbf{O}(3)$.

In Section 3 we find the generators of the $L-$invariant polynomials on S^2 for subgroups $L = \mathbb{T}, \mathbb{O}, \mathbb{T} \oplus \mathbb{Z}_2^c, \mathbb{O}^-, \mathbb{O} \oplus \mathbb{Z}_2^c$, $\mathbb{I}$, and $\mathbb{I} \oplus \mathbb{Z}_2^c$ of $\mathbf{O}(3)$. Here we denote by $\mathbb{Z}_2^c$ the subgroup $\mathbb{Z}_2^c := <-\mathbb{1}> = \{\pm\mathbb{1}\}$ of $\mathbf{O}(3)$. The invariant polynomials will be used to construct equivariant mappings. Furthermore, the generators of the equivariant mappings are studied as well.

For the subsequent discussion it will be of high interest, whether or not there are polynomials having precisely $\mathbb{T}$ symmetry (in the sense that they cannot be written as a sum of polynomials being more symmetric). We resolve this question in Section 4.1. Moreover, we find for each nonplanar subgroup of $\mathbf{O}(3)$ the ring of invariant polynomials and the module of equivariant polynomial mappings in terms of generators and Poincaré–series. In Theorem 4.8 and 4.11 we characterize a complement of $\mathbb{O}\oplus\mathbb{Z}_2^c-$ and $\mathbb{I}\oplus\mathbb{Z}_2^c-$invariant polynomials and show that its dimension is given by a Poincaré–series as well. Similar studies are also given for the equivariants.

Afterwards, in Section 5, we investigate the sets

$$Fix_{(L,G/H)} := \bigcup_{L^* \neq L' \subset L} Fix_{G/H}(L') \tag{1.14}$$

in the cases $H = \mathbf{O}(2) \oplus \mathbb{Z}_2^c$ and $H = \mathbf{O}(2)^-$. Here we denote by $L^* = \{\gamma \in L \mid \gamma y = y \text{ for all } y \in G/H\}$ the stabilizer of this action. Moreover, we look for parametrizations of the connections of $(L, G/H)$.

In Section 6 we introduce a set of basically possible flows (called 'basic flows'), we have found by using the flow formula for different symmetry-breaking terms of the form $h : L^2(S^2) \to L^2(S^2)$, $u \mapsto p \cdot \Theta(u)$, where $\Theta : \mathbb{R} \to \mathbb{R}$ is a smooth function and $p \in C(S^2)$ is a polynomial on S^2 which is invariant under L for some finite supergroup L of $\mathbb{T}$. Using this kind of perturbations, we find lots of heteroclinic orbits for the perturbed flow. However, since this perturbed problem is still of variational structure, it admits no heteroclinic cycle.

We will overcome this lack in Section 7, when we consider $\mathbb{T}-$equivariant perturbations $h : D \subset L^2(S^2) \to L^2(S^2)$, $u \mapsto e\nabla u$, with some $\mathbb{T}-$equivariant polynomial mapping e. Here and, moreover, in some special cases $e = q \cdot \nabla p$, with both q and p $\mathbb{T}-$invariant, we establish heteroclinic cycles. The special cases are of particular interest because they can be viewed as a perturbation of the diffusion term.

In Section 8, we give some hints how these ideas work out for systems and finally, in the appendix we give some more details on the calculation program which derived most of these flows.

2 The Flow Formula

The aim of this section is to find more information about $L-$equivariant flows restricted to connections of $(L, G/H)$. However, we do not want to discuss that topic in general, as we did it in [11]. Here we are particularly interested in the flow on $M^{\varepsilon,\alpha}$ for $|\alpha| \neq 0$ and $\varepsilon > 0$ small and fixed. $M^{\varepsilon,\alpha}$ as well as $\mathcal{O}(u_\alpha)$ and $\mathcal{O}(u^*)$ are all diffeomorphic to G/H. Our program will therefore be to approximate the manifold $M^{\varepsilon,\alpha}$ by $\mathcal{O}(u^*)$ and, moreover, to find information about the flow on the connections of $(L, M^{\varepsilon,\alpha})$ in terms of quantities which can be calculated on $\mathcal{O}(u^*)$.

As before, we denote by $(u_\alpha, \lambda_\alpha)$, $|\alpha|$ small, the branch of equilibria of (1.1). We assume that at $(u, \lambda) = (0, \lambda_0)$ the center manifold theorem is applicable (cf. [6], 6.3, for growth conditions on the nonlinearity and [2] for the handling of the parameter λ). This gives

$$u_\alpha = \alpha u^* + \sigma(\alpha u^*, \lambda_\alpha), \tag{2.1}$$

with a smooth function $\sigma : \ker A(\lambda_0) \times \mathbb{R} \to \ker A(\lambda_0)^\perp \subset L^2(S^2)$, which has the properties

$$\sigma(0,\lambda)=0 \quad \text{for all } \lambda \text{ and } \quad D_1\sigma(0,\lambda_0)=0. \tag{2.2}$$

In order to calculate the flow on connections of $(L, M^{\varepsilon,\alpha})$ in a first approximation, it is necessary to have a parametrization of these connections. However, the manifold $M^{\varepsilon,\alpha}$ is not so easy to handle and therefore we look for better realizations of G/H. For this reason let

$$V := \ker A(\lambda_0) \subset L^2(S^2). \tag{2.3}$$

Since $A(\lambda_0)$ is assumed to be $G-$equivariant, it follows that V is $G-$invariant and hence the action of G on $L^2(S^2)$ restricts to V, i.e. we have $G\times V \to V$, $(\gamma, v)\mapsto \gamma v$.

In (1.5) we assumed that both $u^* \in \ker A(\lambda_0)$ and u_α have isotropy subgroup H. Therefore a realization of G/H which is (after rescaling) a good approximation of the group orbits $\mathcal{O}(u_\alpha)$, for $|\alpha|\neq 0$ small, is given by

$$G/H \cong \mathcal{O}(u^*) = \{\gamma u^* | \gamma\in G\} \subset \ker A(\lambda_0) \subset L^2(S^2). \tag{2.4}$$

We thus have three different realizations of G/H, namely $M^{\varepsilon,\alpha}$, $\mathcal{O}(u_\alpha)$, and $\mathcal{O}(u^*)$ which are all $L-$equivariantly diffeomorphic.

Assume $\Upsilon \in \mathcal{H}_{(L,G/H)}$ is a connection of $(L, G/H)$ connecting two equilibria $e_1, e_2 \in \mathcal{E}_{(L,G/H)}$. In particular Υ is contained in some component of the fixed–point subspace $Fix_{G/H}(L')$, $L'\subset L\subset G$, diffeomorphic to $S^1\subset G/H$. Considering again $\mathcal{O}(u^*)$ as a realization of G/H we can parameterize $\Upsilon\subset G/H$ as a part of $\mathcal{O}(u^*)$ explicitly: There exists a smooth function $\gamma^* : \mathbb{R}/2\pi \to G$ such that

$$\omega : \mathbb{R}/2\pi \to S^1 \subset \mathcal{O}(u^*) \subset L^2(S^2), \quad \omega(\varphi) := \gamma^*(\varphi)u^* \tag{2.5}$$

is a nondegenerate parametrization of this S^1 of the above fixed–point subspace, with

$$\Upsilon = \{\omega(\varphi) | \varphi \in (0,\varphi^*)\}, \quad 0<\varphi^*\leq 2\pi, \quad \omega(0)=e_1 \text{ and } \omega(\varphi^*)=e_2. \tag{2.6}$$

Corresponding to ω, the quantity $\tau : \mathbb{R}/2\pi \to \mathbb{R}$

$$\tau(\varphi) := \int_{S^2} \mathfrak{T}(\varphi)\cdot h(\omega(\varphi))dS, \quad \text{with } \mathfrak{T}(\varphi) := \frac{\frac{d}{d\varphi}\omega(\varphi)}{\|\frac{d}{d\varphi}\omega(\varphi)\|} \in \ker A(\lambda_0) \subset L^2(S^2) \tag{2.7}$$

which is the tangent vector on this S^1, will play a crucial role in the following. We introduce similar quantities on $\mathcal{O}(u_\alpha)$. Letting

$$\omega_\alpha(\varphi) := \gamma^*(\varphi)u_\alpha = \alpha\omega(\varphi) + \gamma^*(\varphi)\sigma(\alpha u^*, \lambda_\alpha), \tag{2.8}$$

we find that

$$\{\omega_\alpha(\varphi), \varphi \in \mathbb{R}/2\pi\} \cong S^1 \subset \mathcal{O}(u_\alpha) \tag{2.9}$$

is a parametrization of the S^1 part in the fixed–point subspace $Fix_{\mathcal{O}(u_\alpha)}(L')$, and

$$\Upsilon_\alpha := \{\omega_\alpha(\varphi) | \varphi \in (0, \varphi^*)\} \tag{2.10}$$

is a connection of $(L, \mathcal{O}(u_\alpha))$. Similarly,

$$\tau_\alpha(\varphi) := \int_{S^2} \mathfrak{T}_\alpha(\varphi) \cdot h(\omega_\alpha(\varphi)) dS, \quad \text{with } \mathfrak{T}_\alpha(\varphi) := \frac{\frac{d}{d\varphi}\omega_\alpha(\varphi)}{\|\frac{d}{d\varphi}\omega_\alpha(\varphi)\|} \subset L^2(S^2) \tag{2.11}$$

is defined. Once we add a symmetry-breaking perturbation term as in (1.8), we know already that the invariant manifold $\mathcal{O}(u_\alpha)$ of (1.1) gets slightly perturbed to $M^{\varepsilon,\alpha}$, an invariant manifold of (1.8), which is $L-$equivariantly diffeomorphic to $\mathcal{O}(u_\alpha)$. Let

$$\rho_{\varepsilon,\alpha} : \mathcal{O}(u_\alpha) \to M^{\varepsilon,\alpha} \tag{2.12}$$

denote this $L-$equivariant diffeomorphism with $\rho_{0,\alpha} = \mathrm{Id}$. Now

$$\tilde{\omega}_{\varepsilon,\alpha}(\varphi) := \rho_{\varepsilon,\alpha}(\omega_\alpha(\varphi)) \tag{2.13}$$

gives a parametrization of

$$\tilde{\Upsilon}_{\varepsilon,\alpha} := \{\tilde{\omega}_{\varepsilon,\alpha}(\varphi) | \varphi \in (0, \varphi^*)\}, \tag{2.14}$$

which is a connection of $(L, M^{\varepsilon,\alpha})$ due to the $L-$equivariance of $\rho_{\varepsilon,\alpha}$. In particular it is a one-dimensional invariant manifold of the flow generated by (1.8). Both $\tilde{\omega}_{\varepsilon,\alpha}(0)$ and $\tilde{\omega}_{\varepsilon,\alpha}(\varphi^*)$ are equilibria of $(L, M^{\varepsilon,\alpha})$, and therefore also equilibria for the flow in (1.8) (cf. [12], Proposition 1.6).

However, the flow on $\tilde{\Upsilon}_{\varepsilon,\alpha}$ is by no means clear, although the flow for the unperturbed problem on Υ_α was trivial. Indeed, it will turn out that we will obtain nontrivial flows in particular cases.

For the following development we use that the direction of the flow on a one-dimensional invariant manifold can be obtained by the inner product of the tangent vector and the vector field. To be precise:

Remark 2.1 *Let $M \subset L^2(S^2)$ be a one-dimensional invariant manifold for the flow $\Phi : \mathbb{R} \times L^2(S^2) \to L^2(S^2)$. Then $w \in M$ is an equilibrium for the flow if and only if*

$$\int_{S^2} \mathfrak{T}(w) \cdot \frac{d}{dt}(\Phi_t(w))_{|t=0}\, dS = 0, \tag{2.15}$$

where $\mathfrak{T}(w) \in L^2(S^2)$ denotes a tangent vector on M at the point w.
Hence in order to determine whether $\tilde{\omega}_{\varepsilon,\alpha}(\varphi) \in \tilde{\Upsilon}_{\varepsilon,\alpha}$ is an equilibrium, we have to calculate

$$\tilde{\tau}_{\varepsilon,\alpha}(\varphi) := \int_{S^2} \mathfrak{T}_{\varepsilon,\alpha}(\varphi) \cdot \frac{d}{dt}(\tilde{\Phi}_t^{\varepsilon,\lambda_\alpha}(\tilde{\omega}_{\varepsilon,\alpha}(\varphi)))_{|t=0}\, dS, \quad \text{with } \mathfrak{T}_{\varepsilon,\alpha}(\varphi) := \frac{\frac{d}{d\varphi}\tilde{\omega}_{\varepsilon,\alpha}(\varphi)}{\|\frac{d}{d\varphi}\tilde{\omega}_{\varepsilon,\alpha}(\varphi)\|} \subset L^2(S^2), \tag{2.16}$$

where again $\tilde{\Phi}^{\varepsilon,\lambda_\alpha}$ denotes the flow generated by (1.8). The following theorem due to Lauterbach and Roberts (cf. [13]) decides for sufficiently small $|\alpha| \neq 0$ and $\varepsilon > 0$ the sign of $\tilde{\tau}_{\varepsilon,\alpha}(\varphi)$. Therefore the direction of the perturbed flow on the connections $\tilde{\Upsilon}_{\varepsilon,\alpha}$ can be calculated. In particular heteroclinic orbits on $\tilde{\Upsilon}_{\varepsilon,\alpha}$ can be established.

Theorem 2.2 *Consider two closed subgroups L and H of $G = \mathbf{SO}(3)$ or $\mathbf{O}(3)$ and the $G-$equivariant semi-dynamical system generated by (1.1) near a bifurcation point $(0, \lambda_0) \in L^2(S^2) \times \mathbb{R}$ of (1.4). We assume that $\ker A(\lambda_0) \subset L^2(S^2)$ is nontrivial and $u^* \in \ker A(\lambda_0)$ has isotropy subgroup H. Moreover, a branch of equilibria with isotropy subgroup H as in (2.1) is assumed to exist. Let the connections $\Upsilon \subset \mathcal{O}(u^*)$ and $\Upsilon_\alpha \subset \mathcal{O}(u_\alpha)$ of $(L, \mathcal{O}(u^*))$ and $(L, \mathcal{O}(u_\alpha))$ be given (see (2.6) and (2.10)).*

We perturb the flow of (1.1) by an $L-$equivariant smooth mapping $h : D \subset L^2(S^2) \to L^2(S^2)$ which is homogeneous of order μ, i.e.

$$h(\alpha u) = \alpha^\mu h(u), \text{ for all } \alpha > 0 \text{ and } u \in D. \tag{2.17}$$

Then for sufficiently small $|\alpha| \neq 0$ and $\varepsilon > 0$ there is a one-dimensional invariant manifold $\tilde{\Upsilon}_{\varepsilon,\alpha} \subset M^{\varepsilon,\alpha} \subset L^2(S^2)$ for the perturbed $L-$equivariant semi-dynamical system (1.8) and the direction of the flow at $\tilde{\omega}_{\varepsilon,\alpha}(\varphi)$ is determined by $\tilde{\tau}_{\varepsilon,\alpha}(\varphi)$ (see (2.13) and (2.16)). The sign of $\tilde{\tau}_{\varepsilon,\alpha}(\varphi)$ is given by $\tau(\varphi)$ (see (2.7)) in the following sense:

1. $\forall \delta > 0\ \exists \alpha_0 > 0$, such that $\forall \alpha \in [-\alpha_0, \alpha_0]\backslash\{0\}\ \exists \varepsilon_0 = \varepsilon_0(\alpha) > 0$ with

$$|\tau(\varphi)| \geq \delta,\ \varphi \in \mathbb{R}/2\pi \Rightarrow \operatorname{sign}(\tilde{\tau}_{\varepsilon,\alpha}(\varphi)) = \operatorname{sign}(\tau(\varphi)), \forall \varepsilon \in (0, \varepsilon_0]. \tag{2.18}$$

2. Let $\varphi_1 \in \mathbb{R}/2\pi$ with $\tau(\varphi_1) = 0$ and $\tau'(\varphi_1) \neq 0$ be given. Then $\exists \alpha_1 > 0$ and $\forall \alpha \in [-\alpha_1, \alpha_1]\backslash\{0\}\ \exists \varepsilon_1 = \varepsilon_1(\alpha) > 0$ such that for all $\varepsilon \in (0, \varepsilon_1]$ there exists a unique zero of $\tilde{\tau}_{\varepsilon,\alpha}$ near φ_1, called $\varphi_{\varepsilon,\alpha}$:

$$\tilde{\tau}_{\varepsilon,\alpha}(\varphi_{\varepsilon,\alpha}) = 0 \text{ and } \tilde{\tau}'_{\varepsilon,\alpha}(\varphi_{\varepsilon,\alpha}) \neq 0. \tag{2.19}$$

Proof. By the above discussion the only thing left to show is that $\tilde{\tau}_{\varepsilon,\alpha}$ can be approximated by τ in the stated sense. Let w.l.o.g. $\alpha > 0$. Essentially, one has to prove that in the topology of $C^1(\mathbb{R}/2\pi)$

$$\frac{\tau_\alpha}{\alpha^\mu} \to \tau \quad \text{as} \quad \alpha \searrow 0 \tag{2.20}$$

and for $\alpha > 0$ fixed

$$\frac{1}{\varepsilon}\tilde{\tau}_{\varepsilon,\alpha} \to \tau_\alpha \quad \text{as} \quad \varepsilon \searrow 0. \tag{2.21}$$

For the proof of (2.20) it is essential to have that h is homogeneous. (2.21) is proved by expanding (2.13):

$$\tilde{\omega}_{\varepsilon,\alpha}(\varphi) = \omega_\alpha(\varphi) + \varepsilon z_\alpha(\varphi) + o(\varepsilon), \text{ as } \varepsilon \searrow 0.$$

It follows

$$\frac{1}{\varepsilon}\tilde{\tau}_{\varepsilon,\alpha}(\varphi) - \tau_\alpha(\varphi) = \int_{S^2} \mathfrak{T}_\alpha(\varphi) \cdot D_u g(\omega_\alpha(\varphi), \lambda_\alpha)[z_\alpha(\varphi)]dS + o(1),$$

as $\varepsilon \searrow 0$. The above integral, however is zero, because of the symmetry of $A(\lambda)$ and since $\mathfrak{T}_\alpha(\varphi) \in \ker(D_u g(\omega_\alpha(\varphi), \lambda_\alpha))$. The details will be given in [13]. □

It is remarkable, that the flow direction depends on $u^* \in \ker A(\lambda_0)$ and therefore on the representation of the group action of G on $\ker A(\lambda_0)$ (see also Section 6 for more details).

Remark 2.3 *In case $\tau = \tau(\varphi), \varphi \in \mathbb{R}/2\pi$, is a function having only simple zeros, the same is true for $\tilde{\tau}_{\varepsilon,\alpha}$ for $|\alpha| \neq 0$ and $\varepsilon > 0$ sufficiently small.*

Remark 2.4 *In the sequel we will calculate instead of $\tau(\varphi)$ only the 'flow formula'*

$$\mathcal{F}^h_\Upsilon(\varphi) := \int_{S^2} \bar{\mathfrak{T}}(\varphi) \cdot h(\omega(\varphi))dS, \quad \varphi \in \mathbb{R}/2\pi, \tag{2.22}$$

with $\bar{\mathfrak{T}}(\varphi) := \frac{d}{d\varphi}\omega(\varphi)$, since sign and simple zeros of $\mathcal{F}^h_\Upsilon$ and τ are the same.

Remark 2.5 *In case we use $L-$equivariant perturbations $\bar{h} : D \subset L^2(S^2) \to L^2(S^2)$ of the form*

$$\bar{h}(u) = h(u) + o(\|u\|^{\mu+1}), \text{ as } u \to 0,$$

with h as in (2.17), we find that Theorem 2.2 is applicable to $\bar{h}$, too. The flow direction for $\bar{h}$ is the same as the one for h.

3 The Invariants and Equivariants of the Exceptional Subgroups of O(3)

As mentioned before, we want to restrict ourselves to symmetry-breaking terms which have at least $\mathbb{T}$ symmetry. Actually we discuss the cases $L = \mathbb{T}, \mathbb{T}\oplus\mathbb{Z}_2^c, \mathbb{O}, \mathbb{O}^-, \mathbb{O}\oplus\mathbb{Z}_2^c, \mathbb{I}$ and $\mathbb{I}\oplus\mathbb{Z}_2^c$ in detail. Note again that $\mathbb{Z}_2^c =< -\mathbb{1} >\subset \mathbf{O}(3)$. Some elementary facts on these groups might be found for instance in [1]. In order to understand the effects for a large number of perturbations, we first classify possible perturbation terms. This classification is based on invariant theory. An important tool is the so called Poincaré–series (see [15, 16]). It is defined as

$$P_{\mathcal{R}}^L(t) = \sum_{d=0}^{\infty}(dim_{\mathbb{C}}(\mathcal{R}_d^L)) \cdot t^d \tag{3.1}$$

where $\mathcal{R}_d^L$ is the space of $L-$invariant homogeneous polynomials of degree d. A well known result (cf. e.g. [16] Proposition 4.1.3) gives a method how to calculate the Poincaré–series for a finite group L:

$$P_{\mathcal{R}}^L(t) = \frac{1}{|L|}\sum_{\gamma\in L}\det(\mathbb{1} - t\cdot\gamma)^{-1}, \tag{3.2}$$

In case of a compact Lie group, the sum has to be replaced by the Haar integral. We refer to (3.1) as the Poincaré–series for the algebra of invariant polynomials.

A similar formula is true for the module of equivariant mappings. Let $\mathcal{M}^L$ denote the module of L–equivariant polynomial mappings, we define the Poincaré–series for this module as

$$P_{\mathcal{M}}^L(t) = \sum_{d=0}^{\infty}(dim_{\mathbb{C}}(\mathcal{M}_d^L)) \cdot t^d, \tag{3.3}$$

where $\mathcal{M}_d^L$ denotes the subspace of those mappings having degree d. This series can be represented as

$$P_{\mathcal{M}}^L(t) = \frac{1}{|L|}\sum_{\gamma\in L}\frac{\overline{\chi}(\gamma)}{\det(\mathbb{1} - t\gamma)}. \tag{3.4}$$

We would like to point out, that although these formulas are proved in the complex case they also apply to the real case as well.

3.1 Generators for the Algebra of Invariant Polynomials

In this section we look at the natural representations of the exceptional subgroups of $\mathbf{O}(3)$ on $\mathbb{R}^3$ and determine the generators of the algebra of invariant functions and the module of equivariant polynomial mappings, respectively. Of course the generating set is not unique, we just present one choice of generators, which prove to be useful for the application we have in mind.

3.1.1 The Invariants for the Action of $\mathbb{T}$

The Poincaré–Series. The Poincaré–series for the three dimensional representation of $\mathbb{T}$ is given by

$$\begin{aligned} P_{\mathcal{R}}^{\mathbb{T}}(t) &= \frac{1}{12}\left(\frac{1}{(1-t)^3} + \frac{3}{(1-t)(1+t)^2} + \frac{8}{(1-t)(1+t+t^2)}\right) \\ &= \frac{1-t^2+t^4}{(1-t^2)^2(1-t^3)} \\ &= \frac{1+t^6}{(1-t^2)(1-t^3)(1-t^4)}. \end{aligned}$$

It is well known that the ring of invariants is Cohen-Macauley [17]. It can be written as a free module over the primary invariants. Since the representation of the Poincaré–series in terms of rational functions is not unique, the validity of the following representation is shown by giving a set of algebraically independent generators with the respective degrees. This remark applies to all computations of Poincaré–series in this paper. The interpretation is as follows: we expect four generators of the ring of invariant polynomials: $I_2^{\mathbb{T}}$, $I_3^{\mathbb{T}}$, $I_4^{\mathbb{T}}$, $I_6^{\mathbb{T}}$, where the first three form an algebraically independent set. The last one is not in the ring generated by $I_2^{\mathbb{T}}, I_3^{\mathbb{T}}, I_4^{\mathbb{T}}$, but it satisfies an algebraic relation, i.e. there exists a polynomial $a : \mathbb{R}^4 \to \mathbb{R}$ with $a(I_2^{\mathbb{T}}, I_3^{\mathbb{T}}, I_4^{\mathbb{T}}, I_6^{\mathbb{T}}) = 0$ (see (3.6)).

The Invariant Polynomials. The group action on $\mathbb{R}^3$ is as follows: the elements of order two send two variables to their respective negatives, one element of order three gives cyclic permutation of the variables x, y, z. For the sequel we shall fix our attention on this $\mathbb{T}$ subgroup of $\mathbf{O}(3)$. The function $I_2^{\mathbb{T}}(x,y,z) = x^2+y^2+z^2$ is certainly invariant. Since there is (up to multiplication with constants) only one quadratic invariant, $I_2^{\mathbb{T}}$ has the form given. The cubic function xyz is invariant, again by uniqueness $I_3^{\mathbb{T}}(x,y,z) = xyz$. Since $x^4+y^4+z^4$ is invariant and not a multiple of $(I_2^{\mathbb{T}})^2$, we may choose $I_4^{\mathbb{T}}(x,y,z) = x^4+y^4+z^4$. The polynomial $x^6+y^6+z^6$ is obviously invariant under the action. However, it is not linearly independent from the functions generated by $I_2^{\mathbb{T}}, I_3^{\mathbb{T}}$ and $I_4^{\mathbb{T}}$ since

$$x^6+y^6+z^6 = -\frac{1}{2}(I_2^{\mathbb{T}})^3 + \frac{3}{2} I_2^{\mathbb{T}} I_4^{\mathbb{T}} + 3(I_3^{\mathbb{T}})^2. \tag{3.5}$$

The invariant $I_6^{\mathbb{T}}(x,y,z)$ is given by

$$I_6^{\mathbb{T}}(x,y,z) = (x^2-y^2)(x^2-z^2)(y^2-z^2).$$

This polynomial is invariant with respect of any sign change in any of the variables. The rotation which maps $x \to y, y \to z$ and $z \to x$ transforms this function to

$$(y^2-z^2)(y^2-x^2)(z^2-x^2),$$

which equals $I_6^{\mathbb{T}}$. In order to simplify notation we define

$$\begin{gathered} \rho_2(x,y,z) := x^2+y^2+z^2, \quad \rho_4(x,y,z) := x^4+y^4+z^4, \quad \rho_6(x,y,z) := x^6+y^6+z^6, \\ \tau_3(x,y,z) := xyz \quad \text{and} \quad \tau_6(x,y,z) := (x^2-y^2)(x^2-z^2)(y^2-z^2). \end{gathered}$$

Hence a set of generators of the $\mathbb{T}-$invariant polynomials is given by ρ_2, τ_3, ρ_4 and τ_6. The algebraic relation turns out to be

$$\tau_6^2 = -\frac{1}{4}\rho_2^6 + \rho_2^4\rho_4 + 5\rho_2^3\tau_3^2 - \frac{5}{4}\rho_2^2\rho_4^2 - 9\rho_2\tau_3^2\rho_4 - 27\tau_3^4 + \frac{1}{2}\rho_4^3. \tag{3.6}$$

3.1.2 The Invariants of $\mathbb{T}\oplus\mathbb{Z}_2^c$

For the three dimensional representation of $\mathbb{T}\oplus\mathbb{Z}_2^c$ the Poincaré–series is

$$P_{\mathcal{R}}^{\mathbb{T}\oplus\mathbb{Z}_2^c}(t) = \frac{1}{2}(P_{\mathcal{R}}^{\mathbb{T}}(t) + P_{\mathcal{R}}^{\mathbb{T}}(-t)) = \frac{1+t^6}{(1-t^2)(1-t^4)(1-t^6)}.$$

A set of generators of the algebra of $\mathbb{T}\oplus\mathbb{Z}_2^c-$invariant polynomials is given by

$$\rho_2, \rho_4, \rho_6 \text{ and } \tau_6.$$

The first three are algebraically independent. τ_6 is not in the ring generated by the first three, but satisfies an algebraic relation, which is easily derived from (3.5) and (3.6):

$$\tau_6^2 = -\frac{1}{6}\rho_2^6 + \frac{3}{2}\rho_2^4\rho_4 - \frac{4}{3}\rho_2^3\rho_6 - \frac{7}{2}\rho_2^2\rho_4^2 + 6\rho_2\rho_4\rho_6 + \frac{1}{2}\rho_4^3 - 3\rho_6^2. \tag{3.7}$$

3.1.3 The Invariants of $\mathbb{O}$

The Poincaré–series for the three dimensional representation of $\mathbb{O}$ is given by

$$\begin{aligned} P_{\mathcal{R}}^{\mathbb{O}}(t) &= \frac{1}{24}\left(\frac{1}{(1-t)^3}+\frac{9}{(1-t)(1+t)^2}+\frac{8}{(1-t)(1+t+t^2)}+\frac{6}{(1-t)(1+t^2)}\right) \\ &= \frac{1-t^3+t^6}{(1-t^2)(1-t^3)(1-t^4)} = \frac{1+t^9}{(1-t^2)(1-t^4)(1-t^6)}. \end{aligned}$$

There is only one subgroup $\mathbb{O}\subset \mathbf{O}(3)$ with $\mathbb{O}\supset \mathbb{T}$ and the functions which are invariant under $\mathbb{O}$ are obviously also invariant under $\mathbb{T}$. This gives $I_2^{\mathbb{O}}=\rho_2$ and $I_4^{\mathbb{O}}=\rho_4$. In addition to the elements in $\mathbb{T}$ we get an action $x\to y,\ y\to -x,\ z\to z$ of an element of order 4. The function τ_6 is not invariant under this action. However the function ρ_6 is invariant. In this case it is not in the span of $(I_2^{\mathbb{O}})^3,\ I_2^{\mathbb{O}}I_4^{\mathbb{O}}$. Therefore

$$I_6^{\mathbb{O}}(x,y,z)=x^6+y^6+z^6.$$

Observe that the element of order 4 in $\mathbb{O}$ changes the sign of τ_3 and τ_6. Therefore the product is invariant under $\mathbb{O}$ and the set of generators is given by ρ_2,ρ_4,ρ_6 and $\tau_3\cdot\tau_6$. The algebraic relation is obvious from (3.5) and (3.7).

3.1.4 The Invariants of $\mathbb{O}^-$

The Poincaré–series can be computed considering the elements in $\mathbb{T}$ and outside $\mathbb{T}$ separately. We obtain

$$\frac{1}{24}\left(\frac{1}{(1-t)^3}+\frac{3}{(1-t)(1+t)^2}+\frac{8}{(1-t)(1+t+t^2)}+\frac{6}{(1-t)^2(1+t)}+\frac{6}{(1+t)(1+t^2)}\right).$$

One finds

$$\begin{aligned} P_{\mathcal{R}}^{\mathbb{O}^-}(t) &= \frac{1}{(1-t)^3(1+t)^2(1+t+t^2)(1+t^2)} \\ &= \frac{1}{(1-t)(1+t)(1-t^2)(1-t^3)(1+t^2)} = \frac{1}{(1-t^2)(1-t^3)(1-t^4)}. \end{aligned}$$

The generators of the $\mathbb{O}^-$–invariant polynomials are given by

$$I_2^{\mathbb{O}^-}=\rho_2, I_3^{\mathbb{O}^-}=\tau_3,\ I_4^{\mathbb{O}^-}=\rho_4.$$

3.1.5 The Invariants of $\mathbb{O}\oplus\mathbb{Z}_2^c$

In this case the Poincaré–series is given by

$$P_{\mathcal{R}}^{\mathbb{O}\oplus\mathbb{Z}_2^c}(t) = \frac{1}{2}(P_{\mathcal{R}}^{\mathbb{O}}(t) + P_{\mathcal{R}}^{\mathbb{O}}(-t)) = \frac{1}{(1-t^2)(1-t^4)(1-t^6)}.$$

Comparing this series with the one of $\mathbb{O}$ and $\mathbb{O}^-$ tells us that the functions of order 6 which are invariant under $\mathbb{O}$, $\mathbb{O}^-$ and $\mathbb{O}\oplus\mathbb{Z}_2^c$ are all the same. The tetrahedral group has an extra fixed function which is not fixed under either of these groups, namely τ_6. The generators of the $\mathbb{O}\oplus\mathbb{Z}_2^c$–invariant polynomials are ρ_2, ρ_4 and ρ_6.

3.1.6 The Invariants of $\mathbb{I}$

The Poincaré–Series. We begin again by computing the Poincaré–series.

$$\begin{aligned}
P_{\mathcal{R}}^{\mathbb{I}}(t) &= \frac{1}{60}\left(\frac{1}{(1-t)^3} + \frac{15}{(1-t)(1+t)^2} + \frac{20}{(1-t)(1+t+t^2)}\right. \\
&+ \left.\frac{12}{(1-t)(1-2(\cos(2\pi/5))t+t^2)} + \frac{12}{(1-t)(1-2(\cos(4\pi/5))t+t^2)}\right) \\
&= \frac{1}{60}\left(\frac{1}{(1-t)^3} + \frac{15}{(1-t)(1+t)^2} + \frac{20}{(1-t)(1+t+t^2)} + \frac{12(2+t+2t^2)}{(1-t)(1+t+t^2+t^3+t^4)}\right) \\
&= \frac{1+t-t^3-t^4-t^5+t^7+t^8}{(1-t)^3(1+t)^2(1+t+t^2)(1+t+t^2+t^3+t^4)} \\
&= \frac{1+t-t^3-t^4-t^5+t^7+t^8}{(1+t)(1-t^2)(1-t^3)(1-t^5)} = \frac{(1+t-t^3-t^4-t^5+t^7+t^8)(1-t+t^2)}{(1+t)(1-t^3)(1-t+t^2)(1-t^2)(1-t^5)} \\
&= \frac{1-t^5+t^{10}}{(1-t^6)(1-t^2)(1-t^5)} = \frac{1+t^{15}}{(1-t^2)(1-t^6)(1-t^{10})}.
\end{aligned}$$

The Invariant Polynomials. In this case it is not obvious how to get a complete set of generators of the algebra of $\mathbb{I}$–invariant polynomials. It is clear that we still have ρ_2. Furthermore, the supergroup $\mathbb{I} \supset \mathbb{T}$ (with $\mathbb{T}$ fixed as before – cf. 3.1.1) is no longer unique. It will be determined uniquely by any of its $\mathbb{Z}_5$ subgroups, or equivalently, by the rotation axis of this $\mathbb{Z}_5$. There are two different possibilities. To see this consider the projection of the edges of the icosahedron to the unit sphere. This will divide the unit sphere into 20 congruent equilateral triangles. The length of one edge of such an triangle is

$$l_\Delta = \arccos\left(\frac{\sqrt{5}}{5}\right).$$

The first rotation axis d_1 of $\mathbb{Z}_5 \subset \mathbb{I}$ is obtained by rotating the $x-$axis by the angle $\frac{1}{2}l_\Delta$ in direction of the $z-$axis (cf. Figure 8 for a geometrical illustration):

$$B := \begin{pmatrix} \cos(\frac{1}{2}l_\Delta) & 0 & -\sin(\frac{1}{2}l_\Delta) \\ 0 & 1 & 0 \\ \sin(\frac{1}{2}l_\Delta) & 0 & \cos(\frac{1}{2}l_\Delta) \end{pmatrix}, \quad d_1 = B \begin{pmatrix} 1 \\ 0 \\ 0 \end{pmatrix} = \begin{pmatrix} \sqrt{\frac{1}{2}+\frac{\sqrt{5}}{10}} \\ 0 \\ \sqrt{\frac{1}{2}-\frac{\sqrt{5}}{10}} \end{pmatrix}.$$

Similar, we find another icosahedral supergroup of $\mathbb{T}$, which we will denote by $\tilde{\mathbb{I}}$, as $\tilde{\mathbb{I}} :=< \mathbb{T}, \tilde{\mathbb{Z}}_5 >$, where the axis of rotation for this $\tilde{\mathbb{Z}}_5$ subgroup is obtained by rotating the $x-$axis by the angle $\frac{1}{2}l_\Delta$ in direction of the $y-$axis: $d_2 := (\sqrt{\frac{1}{2}+\frac{\sqrt{5}}{10}}, \sqrt{\frac{1}{2}-\frac{\sqrt{5}}{10}}, 0)$. Again, from Figure 8 it is not difficult to see, that any other cyclic subgroup of order 5 in a icosahedral group, which contains $\mathbb{T}$ is conjugate to either $\mathbb{Z}_5$ or $\tilde{\mathbb{Z}}_5$.

Proposition 3.1 *There is a set of generators of the algebra of $\mathbb{I}-$invariant polynomials containing ρ_2 and the following elements:*

$$\begin{aligned} \iota_6 &:= \tau_6 + \sqrt{5}\left(-\frac{1}{3}\rho_2^3 + \rho_2\rho_4 - \frac{11}{15}\rho_6\right) \\ \iota_{10} &:= \rho_4\tau_6 + \sqrt{5}\left(\frac{26}{9}\rho_2^3\rho_4 - \frac{64}{45}\rho_2^2\rho_6 - 3\rho_2\rho_4^2 + \frac{19}{9}\rho_4\rho_6\right) \\ \text{and } \iota_{15} &:= \tau_3\tau_6\left(\frac{56}{145}\rho_2^3 - \frac{39}{29}\rho_2\,\rho_4 + \rho_6\right) + \sqrt{5}\tau_3\left(\frac{199}{2900}\rho_2^6 - \frac{1383}{2900}\rho_2^4\rho_4\right. \\ &\quad \left. + \frac{326}{725}\rho_2^3\rho_6 + \frac{69}{100}\rho_2^2\rho_4^2 - \frac{972}{725}\rho_2\,\rho_4\,\rho_6 + \frac{27}{116}\rho_4^3 + \frac{279}{725}\rho_6^2\right). \end{aligned}$$

The algebraic relation is

$$\begin{aligned} \iota_{15}^2 &= \left(-\frac{380057}{15138000}\rho_2^{15} - \frac{33999}{1682000}\rho_2^9\iota_6^2 + \frac{99}{8410}\rho_2^3\iota_6^4 + \frac{17397}{210250}\rho_2^7\iota_6\,\iota_{10} - \frac{243}{6728}\rho_2\,\iota_6^3\iota_{10}\right. \\ &- \left.\frac{59751}{1682000}\rho_2^5\iota_{10}^2\right) + \sqrt{5}\left(-\frac{130367}{5046000}\rho_2^{12}\iota_6 + \frac{7167}{336400}\rho_2^6\iota_6^3 - \frac{243}{33640}\iota_6^5 + \frac{38991}{1682000}\rho_2^{10}\iota_{10}\right. \\ &+ \left.\frac{2187}{336400}\rho_2^4\iota_6^2\iota_{10} - \frac{891}{67280}\rho_2^2\iota_6\,\iota_{10}^2 + \frac{243}{67280}\iota_{10}^3\right). \end{aligned}$$

Finally, a set of generators of the algebra of $\tilde{\mathbb{I}}-$invariants is given by

$$\tilde{\iota}_6 := -\tau_6 + \sqrt{5}\left(-\frac{1}{3}\rho_2^3 + \rho_2\rho_4 - \frac{11}{15}\rho_6\right)$$

$$\begin{aligned}
\tilde{\iota}_{10} &:= -\rho_4\tau_6 + \sqrt{5}\left(\frac{26}{9}\rho_2^3\rho_4 - \frac{64}{45}\rho_2^2\rho_6 - 3\rho_2\rho_4^2 + \frac{19}{9}\rho_4\rho_6\right) \\
\text{and } \tilde{\iota}_{15} &:= \tau_3\tau_6\left(\frac{56}{145}\rho_2^3 - \frac{39}{29}\rho_2\,\rho_4 + \rho_6\right) - \sqrt{5}\tau_3\left(\frac{199}{2900}\rho_2^6 - \frac{1383}{2900}\rho_2^4\rho_4\right. \\
&\quad \left. + \frac{326}{725}\rho_2^3\rho_6 + \frac{69}{100}\rho_2^2\rho_4^2 - \frac{972}{725}\rho_2\,\rho_4\,\rho_6 + \frac{27}{116}\rho_4^3 + \frac{279}{725}\rho_6^2\right).
\end{aligned}$$

Proof. We will first consider the $\mathbb{I}$−invariants; the $\tilde{\mathbb{I}}$ case then follows easily. Any of the above given polynomials is $\mathbb{T}$−invariant by construction. To show $\mathbb{I}$−invariance, it suffices to show the invariance under $\mathbb{Z}_5 \subset \mathbb{I}$, or, equally well, under a generating element ξ_5 of this $\mathbb{Z}_5$. ξ_5 is a rotation about an angle of $\frac{2}{5}\pi$ around the d_1−axis:

$$\xi_5 = B \begin{pmatrix} 1 & 0 & 0 \\ 0 & \cos(\frac{2}{5}\pi) & -\sin(\frac{2}{5}\pi) \\ 0 & \sin(\frac{2}{5}\pi) & \cos(\frac{2}{5}\pi) \end{pmatrix} B^{-1}.$$

A short calculation gives

$$\xi_5^{-1} = \begin{pmatrix} \frac{1}{4}(1+\sqrt{5}) & \frac{1}{2} & \frac{1}{4}(-1+\sqrt{5}) \\ -\frac{1}{2} & \frac{1}{4}(-1+\sqrt{5}) & \frac{1}{4}(1+\sqrt{5}) \\ \frac{1}{4}(-1+\sqrt{5}) & -\frac{1}{4}(1+\sqrt{5}) & \frac{1}{2} \end{pmatrix}.$$

It remains to check that

$$\iota_6(\xi_5^{-1}\begin{pmatrix} x \\ y \\ z \end{pmatrix}) = (x^2-y^2)(y^2-z^2)(z^2-x^2) - \frac{1}{15}\sqrt{5}(x^6+y^6+z^6) - 2\sqrt{5}x^2y^2z^2 = \iota_6\begin{pmatrix} x \\ y \\ z \end{pmatrix},$$

and similar the equation

$$\xi_5\iota_i\begin{pmatrix} x \\ y \\ z \end{pmatrix} = \iota_i(\xi_5^{-1}\begin{pmatrix} x \\ y \\ z \end{pmatrix}) = \iota_i\begin{pmatrix} x \\ y \\ z \end{pmatrix}$$

holds for $i = 10$ and 15. This requires a little patience, though no real flair, and therefore we leave that and also the verification of the algebraic relation to the reader. Due to the Poincaré–series, we have found all generators of the $\mathbb{I}$−invariant polynomials. To see the generators for $\tilde{\mathbb{I}}$−invariant polynomials, observe that for the two axes of rotation of $\mathbb{Z}_5$ and $\tilde{\mathbb{Z}}_5$

$$d_2 = \xi_4 d_1$$

holds, where ξ_4 is an element of order four in $\mathbb{O} \supset \mathbb{T}$ (which maps $x \to x, y \to -z, z \to y$). Hence, $\tilde{\mathbb{Z}}_5 = \xi_4 \mathbb{Z}_5 \xi_4^{-1}$ and being invariant under $\tilde{\mathbb{I}}$ means being invariant under $\mathbb{T}$ and $\xi_4 \mathbb{Z}_5 \xi_4^{-1}$. As a matter of fact this is the case for $\tilde{\iota}_6, \tilde{\iota}_{10}$ and $\tilde{\iota}_{15}$, because using $\xi_4 \tau_3 = \xi_4^{-1} \tau_3 = -\tau_3$ and $\xi_4 \tau_6 = \xi_4^{-1} \tau_6 = -\tau_6$ it follows

$$\xi_4 \tilde{\iota}_i = \xi_4^{-1} \tilde{\iota}_i = \iota_i, \text{ for } i = 6, 10 \text{ and } 15.$$

$\square$

3.1.7 The Invariants of $\mathbb{I} \oplus \mathbb{Z}_2^c$

Here we have

$$P_{\mathcal{R}}^{\mathbb{I} \oplus \mathbb{Z}_2^c}(t) = \frac{1}{2}(P_{\mathcal{R}}^{\mathbb{I}}(t) + P_{\mathcal{R}}^{\mathbb{I}}(-t)) = \frac{1}{(1-t^2)(1-t^6)(1-t^{10})}.$$

The generators of the $\mathbb{I} \oplus \mathbb{Z}_2^c$–invariant polynomials are $\rho_2, \iota_6\ \iota_{10}$, whereas $\rho_2, \tilde{\iota}_6$ and $\tilde{\iota}_{10}$ generate the $\tilde{\mathbb{I}} \oplus \mathbb{Z}_2^c$–invariants.

3.2 Generators for Modules of Equivariant Polynomial Mappings

3.2.1 The Tetrahedral Equivariants

The Poincaré–Series. For the Poincaré-series for the module of tetrahedral equivariant polynomials we get

$$P_{\mathcal{M}}^{\mathbb{T}}(t) = \frac{t + t^2 + 2t^3 + t^4 + t^5}{(1-t^2)(1-t^3)(1-t^4)} \tag{3.8}$$

and

$$P_{\mathcal{M}}^{\mathbb{T} \oplus Z_2^c}(t) = \frac{t + 2t^3 + 2t^5 + t^7}{(1-t^2)(1-t^4)(1-t^6)}. \tag{3.9}$$

A Generating Set. From the Poincaré-series we find that there is a set of generators containing one linear, quadratic, quartic, quintic and two cubic elements. We write $E_d^{\mathbb{T}}$ for an element of this list of degree d, the second index gives an enumeration of elements having the same degree. Here is a list of generators:

$$\begin{array}{rcl} (x,y,z) & \mapsto & E_1^{\mathbb{T}} = (x,y,z) \\ & \mapsto & E_2^{\mathbb{T}} = (yz, xz, xy) \\ & \mapsto & E_{3a}^{\mathbb{T}} = (xy^2 + xz^2, x^2y + yz^2, x^2z + y^2z) \end{array}$$

$$\begin{aligned} \mapsto \quad & E_{3b}^{\mathbb{T}} = (-xy^2 + xz^2, x^2y - yz^2, -x^2z + y^2z) \\ \mapsto \quad & E_4^{\mathbb{T}} = (y^3z - yz^3, xz^3 - xz^3, x^3y - xy^3) \\ \mapsto \quad & E_5^{\mathbb{T}} = \nabla(\iota_6) \end{aligned}$$

We write $\epsilon_j = E_j^{\mathbb{T}}$ for $j = 1, 2, 4, 5$ and ϵ_{3a} or ϵ_{3b} for $E_{3a}^{\mathbb{T}}$ and $E_{3b}^{\mathbb{T}}$, respectively.

For a list of generators of the $\mathbb{T} \oplus \mathbb{Z}_2^c$ we just have to restrict to the odd members of our list. However some care is required. Any odd $\mathbb{T}$-equivariant mapping has the right equivariance property, however the odd generators do not generate the odd mappings over the ring of invariant functions. For example, the second fifth degree equivariant is given by $\tau_3\epsilon_2$.

3.2.2 The Octahedral Equivariants

Poincaré–Series. Again we start by giving the respective Poincaré-series for $\mathbb{O}$, $\mathbb{O}^-$ and $\mathbb{O} \oplus \mathbb{Z}_2^c$. We have

$$\begin{aligned} P_{\mathcal{M}}^{\mathbb{O}}(t) &= \frac{t + t^3 + t^4 + t^5 + t^6 + t^8}{(1-t^2)(1-t^4)(1-t^6)} \\ P_{\mathcal{M}}^{\mathbb{O}^-}(t) &= \frac{t + t^2 + t^3}{(1-t^2)(1-t^3)(1-t^4)} \\ P_{\mathcal{M}}^{\mathbb{O}\oplus\mathbb{Z}_2^c}(t) &= \frac{t + t^3 + t^5}{(1-t^2)(1-t^4)(1-t^6)}. \end{aligned}$$

The Generators. From the Poincaré-series it is clear that the module of functions equivariant with respect to $\mathbb{O}^-$ is generated by ϵ_1, ϵ_2 and some cubic mapping. It is easy to check, that this cubic mapping is given by

$$(x, y, z) \mapsto \epsilon_{3a}(x, y, z).$$

From the Poincaré-series we conclude that the space of cubic equivariant polynomial mappings is the same for all octahedral groups.

Theorem 3.2 *1. If n is even, then*

$$\mathcal{M}_n^{\mathbb{T}} = \mathcal{M}_n^{\mathbb{O}} \oplus \mathcal{M}_n^{\mathbb{O}^-}.$$

2. For n odd, we find

$$\mathcal{M}_n^{\mathbb{O}} = \mathcal{M}_n^{\mathbb{O}^-} = \mathcal{M}_n^{\mathbb{O}\oplus\mathbb{Z}_2^c}.$$

Proof. It is easily checked that for even n $\mathcal{M}_n^{\mathbb{T}} = \mathcal{M}_n^{\mathbb{O}} + \mathcal{M}_n^{\mathbb{O}^-}$ and the intersection $\mathcal{M}_n^{\mathbb{O}} \cap \mathcal{M}_n^{\mathbb{O}^-} = \{0\}$. In order to show the second assertion, we notice that $\mathcal{M}_n^{\mathbb{O}\oplus\mathbb{Z}_2^c}$ is contained in both $\mathcal{M}_n^{\mathbb{O}}$ and $\mathcal{M}_n^{\mathbb{O}^-}$. From the Poincaré-series we read off that all the dimensions of these three spaces are equal, which shows the result. □

For the equivariants of degree 4 we conclude that

$$\mathcal{M}_4^{\mathbb{T}} = \mathcal{M}_4^{\mathbb{O}} \oplus \mathcal{M}_4^{\mathbb{O}^-}.$$

$\mathcal{M}_4^{\mathbb{T}}$ is generated by $\tau_3 \mathbb{1}$, $\rho_2\epsilon_2$ and ϵ_4. The first two of these are equivariant with respect to $\mathbb{O}^-$, the last one is equivariant with respect to $\mathbb{O}$.

The Poincaré-series indicates a quintic mapping for the groups $\mathbb{O}$ and $\mathbb{O}\oplus\mathbb{Z}_2^c$. One easily checks that this mapping is given by $\tau_3\epsilon_2$.

For degree 6 we find that $\mathcal{M}_6^{\mathbb{O}}$ is given by products of invariant functions and equivariant mappings of lower degree and $\tau_3\epsilon_{3b}$. In a similar fashion we conclude that $\mathcal{M}_8^{\mathbb{O}}$ is given by products of lower order functions and mappings and the new term $\tau_3\epsilon_5$.

3.2.3 The Icosahedral Equivariants

The Poincaré–Series. For the group $\mathbb{I}$ we find the Poincaré–series

$$P_{\mathcal{M}}^{\mathbb{I}}(t) = \frac{t + t^5 + t^6 + t^9 + t^{10} + t^{14}}{(1-t^2)(1-t^6)(1-t^{10})}. \tag{3.10}$$

From this one gets

$$P_{\mathcal{M}}^{\mathbb{I}\oplus\mathbb{Z}_2^c}(t) = \frac{t + t^5 + t^9}{(1-t^2)(1-t^6)(1-t^{10})}. \tag{3.11}$$

The Generators. Here, we restrict our attention to the group $\mathbb{I}\oplus\mathbb{Z}_2^c$. From the Poincaré–series for the invariant functions one can easily conclude that the gradients of the generators of the invariant functions lead to a set of generators for the module of equivariant mappings. I.e. we find

$$\mathcal{M}^{\mathbb{I}\oplus\mathbb{Z}_2^c} = < \epsilon_1, \nabla\iota_6, \nabla\iota_{10} >_{\mathcal{R}^{\mathbb{I}\oplus\mathbb{Z}_2^c}} . \tag{3.12}$$

4 Precisely $\mathbb{T}\oplus\mathbb{Z}_2^c$ Symmetric Polynomials

The question we want to address here is: "Are there any polynomials having precisely tetrahedral symmetry (in the sense that they cannot be written as a sum of polynomials all of them having more symmetry)"? We will answer this question negatively, but we will also see that there are polynomials having precisely $\mathbb{T}\oplus\mathbb{Z}_2^c$ symmetry in the above sense. The importance of this question is based on the fact that octahedral or icosahedral symmetric perturbations always produce additional equilibria in the flow formula. These perturbations moreover rule out hyperbolic heteroclinic cycles. We therefore can accomplish our final goal of finding heteroclinic cycles only with precisely tetrahedral symmetric perturbations.

Let us start with the invariant polynomials. The same question for the equivariant polynomial mappings is addressed in Subsection 4.2.

4.1 Orthogonal Decomposition of $\bar{\mathcal{R}}^{\mathbb{T}}$

Although some of the following linear spaces are already defined, we give them again for convenience.

Definition 4.1

$$\begin{aligned}
\mathcal{R} &:= \{p : \mathbb{R}^3 \to \mathbb{R} \mid p \textit{ is polynomial } \} \\
\mathcal{R}_i &:= \{p \in \mathcal{R} \mid p \textit{ is homogeneous and } \deg(p) = i\} \\
\mathcal{R}_{\leq i} &:= \bigoplus_{j=0}^{i} \mathcal{R}_j = \{p \in \mathcal{R} \mid \deg(p) \leq i\} \\
\mathcal{R}^L &:= \{p \in \mathcal{R} \mid \gamma p = p \textit{ for all } \gamma \in L\}.
\end{aligned}$$

The spaces $\mathcal{R}_i^L$ and $\mathcal{R}_{\leq i}^L$ are defined analogously.

Resuming the results of the last section we know a minimal set of generators for the following $\mathcal{R}^L$:

Corollary 4.2

$$\begin{aligned}
\mathcal{R}^{\mathbb{T}} &= \mathbb{R}[\rho_2, \tau_3, \rho_4, \tau_6], & \mathcal{R}^{\mathbb{T}\oplus\mathbb{Z}_2^c} &= \mathbb{R}[\rho_2, \rho_4, \rho_6, \tau_6] \\
\mathcal{R}^{\mathbb{O}} &= \mathbb{R}[\rho_2, \rho_4, \rho_6, \tau_3 \cdot \tau_6], & \mathcal{R}^{\mathbb{O}^-} &= \mathbb{R}[\rho_2, \tau_3, \rho_4] \\
\mathcal{R}^{\mathbb{O}\oplus\mathbb{Z}_2^c} &= \mathbb{R}[\rho_2, \rho_4, \rho_6], & \mathcal{R}^{\mathbb{I}} &= \mathbb{R}[\rho_2, \iota_6, \iota_{10}, \iota_{15}]
\end{aligned}$$
$$\mathcal{R}^{\mathbb{I}\oplus\mathbb{Z}_2^c} = \mathbb{R}[\rho_2, \iota_6, \iota_{10}].$$

The dimension of $\mathcal{R}_i^L, L = \mathbb{T}, \mathbb{T}\oplus\mathbb{Z}_2^c, \mathbb{O}, \mathbb{O}^-, \mathbb{O}\oplus\mathbb{Z}_2^c, \mathbb{I}$ and $\mathbb{I}\oplus\mathbb{Z}_2^c$, is given by the i-th coefficient of the Poincaré–series $P_{\mathcal{R}}^L$ (cf. Section 3).

Actually, we are only interested in the restrictions of the above polynomials to the sphere S^2. Therefore let

Definition 4.3

$$\bar{\mathcal{R}} := \{\bar{p} : S^2 \to \mathbb{R} \mid \exists p \in \mathcal{R} \textit{ with } p_{|S^2} = \bar{p}\}$$

and similarly, define $\bar{\mathcal{R}}_i, \bar{\mathcal{R}}_{\leq i}, \bar{\mathcal{R}}^L, \bar{\mathcal{R}}_i^L$ *and* $\bar{\mathcal{R}}_{\leq i}^L$ *as linear spaces of the restrictions of the appropriate polynomials.*

We use for instance $\bar{\rho}_6 : S^2 \to \mathbb{R}$ as the restriction of the polynomial ρ_6 to the sphere and the same notation for the other functions. This agreement will be valid for the whole of this section. Later on, however, we will come back to the notation without bars, because then it won't make a difference, whether the functions are defined on S^2 or $\mathbb{R}^3$.

One immediately finds (note that ρ_2 restricted to the sphere is just a constant!):

Corollary 4.4

$$\begin{aligned} \bar{\mathcal{R}}^{\mathbb{T}} &= \mathbb{R}[\bar{\tau}_3, \bar{\rho}_4, \bar{\tau}_6], & \bar{\mathcal{R}}^{\mathbb{T}\oplus\mathbb{Z}_2^c} &= \mathbb{R}[\bar{\rho}_4, \bar{\rho}_6, \bar{\tau}_6] \\ \bar{\mathcal{R}}^{\mathbb{O}} &= \mathbb{R}[\bar{\rho}_4, \bar{\rho}_6, \bar{\tau}_3 \cdot \bar{\tau}_6], & \bar{\mathcal{R}}^{\mathbb{O}^-} &= \mathbb{R}[\bar{\tau}_3, \bar{\rho}_4] \\ \bar{\mathcal{R}}^{\mathbb{O}\oplus\mathbb{Z}_2^c} &= \mathbb{R}[\bar{\rho}_4, \bar{\rho}_6], & \bar{\mathcal{R}}^{\mathbb{I}} &= \mathbb{R}[\bar{\iota}_6, \bar{\iota}_{10}, \bar{\iota}_{15}] \end{aligned}$$
$$\bar{\mathcal{R}}^{\mathbb{I}\oplus\mathbb{Z}_2^c} = \mathbb{R}[\bar{\iota}_6, \bar{\iota}_{10}].$$

Still we have that the sum of $\bar{\mathcal{R}}_i^L, i \in \mathbb{N}$, spans the whole space $\bar{\mathcal{R}}^L$, but the sum is no longer direct. Recognizing that $\bar{\tau}_6^2$ satisfies an algebraic relation similar as τ_6^2 (see (3.6)) one would guess that

$$\bar{\mathcal{Q}}_i^{\mathbb{T}} := \mathrm{Span}\{\bar{\tau}_3^k \bar{\rho}_4^l \bar{\tau}_6^m \mid 3k + 4l + 6m = i \text{ and } k, l \geq 0,\ m \in \{0, 1\}\}, \quad i \geq 0$$

(with $\bar{\mathcal{Q}}_i^{\mathbb{T}} = \{0\}$ in case no such combination of k, l and m exists) would give a proper decomposition of $\bar{\mathcal{R}}^{\mathbb{T}}$. This is indeed the case.

Define $\bar{\mathcal{Q}}_i^L$ similar for the other relevant subgroups L using their generators from Corollary 4.4 and the algebraic relations from Section 3.

Proposition 4.5 *We have for* $L \supset \mathbb{T}$ *and* $j \geq 2$

$$\bar{\mathcal{R}}_{j-2}^L \subset \bar{\mathcal{R}}_j^L \quad \textit{and } \bar{\mathcal{R}}_{j-2}^L \oplus \bar{\mathcal{Q}}_j^L = \bar{\mathcal{R}}_j^L. \tag{4.1}$$

Furthermore, for $j \geq 0$, $\dim \bar{\mathcal{R}}_j^L = \dim \mathcal{R}_j^L$ *holds and*

$$\bigoplus_{\substack{i=0 \\ j-i=0 \bmod 2}}^{j} \bar{\mathcal{Q}}_i^L = \bar{\mathcal{R}}_j^L, \quad \bigoplus_{i=0}^{j} \bar{\mathcal{Q}}_i^L = \bar{\mathcal{R}}_{j-1}^L \oplus \bar{\mathcal{R}}_j^L = \bar{\mathcal{R}}_{\leq j}^L \ (j \geq 1) \textit{ as well as } \bigoplus_{i=0}^{\infty} \bar{\mathcal{Q}}_i^L = \bar{\mathcal{R}}^L. \tag{4.2}$$

The dimension of the spaces $\bar{\mathcal{Q}}_i^L$ *can be obtained by the coefficients of the modified Poincaré–series*

$$P_{\bar{\mathcal{R}}}^L(s) = (1 - s^2) \cdot P_{\mathcal{R}}^L(s).$$

Proof. Firstly, $\bar{\mathcal{R}}^L_{j-2} \subset \bar{\mathcal{R}}^L_j$, because $\bar{p} \in \bar{\mathcal{R}}^L_{j-2}$ implies $\bar{\rho}_2\bar{p} \in \bar{\mathcal{R}}^L_j$. Therefore, $\bar{\mathcal{R}}^L_{j-2} + \bar{\mathcal{Q}}^L_j \subset \bar{\mathcal{R}}^L_j$ by definition. To show "$\supset$" we assume $L = \mathbb{T}$, since things work out similar for the other subgroups. For any $\bar{p} \in \bar{\mathcal{R}}^{\mathbb{T}}_j$ choose some $p \in \mathcal{R}^{\mathbb{T}}_j$ with $p_{|S^2} = \bar{p}$. By Corollary 4.2 and (3.6) p can be uniquely written as

$$\begin{aligned} p &= \sum_{\substack{2i+3k+4l+6m=j\\ m\in\{0,1\}}} \alpha_{i,k,l,m}\rho_2^i\tau_3^k\rho_4^l\tau_6^m \\ &= \sum_{\substack{3k+4l+6m=j\\ m\in\{0,1\}}} \alpha_{0,k,l,m}\tau_3^k\rho_4^l\tau_6^m + \rho_2 \cdot \sum_{\substack{2(i-1)+3k+4l+6m=j-2\\ i\geq 1,\ m\in\{0,1\}}} \alpha_{i,k,l,m}\rho_2^{i-1}\tau_3^k\rho_4^l\tau_6^m =: q_1 + \rho_2 q_2. \end{aligned}$$

Now $\bar{p} = p_{|S^2} = q_{1|S^2} + \bar{\rho}_2 \cdot q_{2|S^2} \in \bar{\mathcal{Q}}^{\mathbb{T}}_j + 1 \cdot \bar{\mathcal{R}}^{\mathbb{T}}_{j-2}$. Furthermore, the sum is direct: $\bar{\mathcal{R}}^{\mathbb{T}}_{j-2} \cap \bar{\mathcal{Q}}^{\mathbb{T}}_j = \{0\}$. For suppose $\bar{p} \in \bar{\mathcal{R}}^{\mathbb{T}}_{j-2} \cap \bar{\mathcal{Q}}^{\mathbb{T}}_j$ is given. Then we can find $\bar{p} = p_{|S^2} = q_{|S^2}$ with

$$p = \sum_{\substack{2i+3k+4l+6m=j-2\\ m\in\{0,1\}}} \alpha_{i,k,l,m}\rho_2^i\tau_3^k\rho_4^l\tau_6^m \text{ and } q = \sum_{\substack{3k+4l+6m=j\\ m\in\{0,1\}}} \beta_{k,l,m}\tau_3^k\rho_4^l\tau_6^m.$$

Since p is homogeneous of degree $j-2$ and q is homogeneous of degree j, we conclude

$$q(x,y,z) = |(x,y,z)|^j\bar{p}\left(\frac{(x,y,z)}{|(x,y,z)|}\right) = |(x,y,z)|^2 p(x,y,z) \text{ for all } (x,y,z) \in \mathbb{R}^3.$$

In other words $q - \rho_2 p = 0$. But this is a linear combination of terms only of the form $\rho_2^i\tau_3^k\rho_4^l\tau_6^m$, with $2i + 3k + 4l + 6m = j$ and $i,k,l \geq 0, m \in \{0,1\}$. These terms are linearly independent (cf. Section 3) and this ensures that all coefficients must be zero, i.e. $\alpha_{i,k,l,m} = 0$ and $\beta_{k,l,m} = 0$. Consequently, $\bar{p} = 0$.

We proceed proving $\dim \bar{\mathcal{R}}^L_j = \dim \mathcal{R}^L_j$ for any exceptional subgroup L of $\mathbf{O}(3)$. Consider the restriction mapping

$$\mathfrak{R} : \mathcal{R}^L_j \to \bar{\mathcal{R}}^L_j, p \mapsto p_{|S^2}.$$

This map is clearly surjective, but it is also injective, because $\mathfrak{R}(p_1) = \mathfrak{R}(p_2)$ implies

$$p_1(x,y,z) = |(x,y,z)|^j\mathfrak{R}(p_1)\left(\frac{(x,y,z)}{|(x,y,z)|}\right) = |(x,y,z)|^j\mathfrak{R}(p_2)\left(\frac{(x,y,z)}{|(x,y,z)|}\right) = p_2(x,y,z)$$

and this claim is proved. The rest is now easy. By repeatedly applying (4.1) we infer $\bigoplus^j_{\substack{i=0\\ j-i=0 \bmod 2}} \bar{\mathcal{Q}}^L_i = \bar{\mathcal{R}}^L_j$. The rest of (4.2) is immediately clear except $\bar{\mathcal{R}}^L_{j-1} \cap \bar{\mathcal{R}}^L_j = \{0\}$.

Assume $\bar{p} \in \bar{\mathcal{R}}^L_{j-1} \cap \bar{\mathcal{R}}^L_j$ is given and take again $\bar{p} = p_{|S^2} = q_{|S^2}$ with $q \in \mathcal{R}^L_{j-1}$ and $p \in \mathcal{R}^L_j$. Like before we find

$$p(x,y,z) = |(x,y,z)|q(x,y,z) \text{ for all } (x,y,z) \in \mathbb{R}^3.$$

If q were not identically zero, then the right hand side would not be polynomial; a contradiction, since p is polynomial. Hence, $\bar{p} = 0$ and (4.2) is proved. The remaining follows from

$$\dim \bar{\mathcal{Q}}^L_j = \dim \bar{\mathcal{R}}^L_j - \dim \bar{\mathcal{R}}^L_{j-2} = \dim \mathcal{R}^L_j - \dim \mathcal{R}^L_{j-2}$$

and exploiting the fact that the j-th coefficient of $P^L_{\mathcal{R}}$ is equal to $\dim \mathcal{R}^L_j$. □

The following theorem is a first step in order to decompose $\bar{\mathcal{R}}^{\mathbb{T}}$ into spaces of more symmetry.

Theorem 4.6 *Let $\mathbb{T} \subset \mathbf{O}(3)$ be fixed as in Section 3 and $\mathbb{O} \supset \mathbb{T}$. Then*

$$\bar{\mathcal{R}}^{\mathbb{T}} = \bar{\mathcal{R}}^{\mathbb{T}\oplus\mathbb{Z}_2^c} \oplus \bar{\tau}_3 \cdot \bar{\mathcal{R}}^{\mathbb{T}\oplus\mathbb{Z}_2^c} \tag{4.3}$$

$$\textit{and } \bar{\mathcal{R}}^{\mathbb{T}\oplus\mathbb{Z}_2^c} = \bar{\mathcal{R}}^{\mathbb{O}\oplus\mathbb{Z}_2^c} \oplus \bar{\tau}_6 \cdot \bar{\mathcal{R}}^{\mathbb{O}\oplus\mathbb{Z}_2^c}. \tag{4.4}$$

Using $U^{\bar{\mathcal{R}}} := \bar{\tau}_6 \cdot \bar{\mathcal{R}}^{\mathbb{O}\oplus\mathbb{Z}_2^c}$ we find a decomposition of $\bar{\mathcal{R}}^{\mathbb{T}}$ in pairwise orthogonal subspaces with respect to $(.,.)_{L^2(S^2)}$:

$$\bar{\mathcal{R}}^{\mathbb{T}} = \bar{\mathcal{R}}^{\mathbb{O}\oplus\mathbb{Z}_2^c} \oplus \bar{\tau}_3 \cdot \bar{\mathcal{R}}^{\mathbb{O}\oplus\mathbb{Z}_2^c} \oplus \bar{\tau}_3\bar{\tau}_6 \cdot \bar{\mathcal{R}}^{\mathbb{O}\oplus\mathbb{Z}_2^c} \oplus U^{\bar{\mathcal{R}}}. \tag{4.5}$$

Proof. We start proving that both decompositions are orthogonal. For an arbitrary polynomial $\bar{q} \in \bar{\mathcal{R}}^{\mathbb{T}\oplus\mathbb{Z}_2^c}$ we claim $(\bar{\tau}_3, \bar{q})_{L^2(S^2)} = 0$. Integration over S^2 is invariant under $\mathbf{O}(3)$, especially under $\gamma := -\mathbb{1} \in \mathbb{T}\oplus\mathbb{Z}_2^c$. We have

$$(\bar{\tau}_3, \bar{q})_{L^2(S^2)} = (\gamma\bar{\tau}_3, \gamma\bar{q})_{L^2(S^2)} = (-\bar{\tau}_3, \bar{q})_{L^2(S^2)} = -(\bar{\tau}_3, \bar{q})_{L^2(S^2)}$$

and the first claim is proved. Observe that this also gives $\bar{\mathcal{R}}^{\mathbb{T}\oplus\mathbb{Z}_2^c} \cap \bar{\tau}_3 \cdot \bar{\mathcal{R}}^{\mathbb{T}\oplus\mathbb{Z}_2^c} = \{0\}$. The orthogonality in (4.4) follows similarly with $\gamma := \xi_4 \in \mathbb{O}$, the generator of a $\mathbb{Z}_4$ subgroup in $\mathbb{O}$. For an arbitrary polynomial $\bar{q} \in \bar{\mathcal{R}}^{\mathbb{O}\oplus\mathbb{Z}_2^c}$ we infer from $\gamma\bar{\tau}_6 = -\bar{\tau}_6$

$$(\bar{\tau}_6, \bar{q})_{L^2(S^2)} = (\gamma\bar{\tau}_6, \gamma\bar{q})_{L^2(S^2)} = -(\bar{\tau}_6, \bar{q})_{L^2(S^2)}$$

and $\bar{\mathcal{R}}^{\mathbb{O}\oplus\mathbb{Z}_2^c} \cap \bar{\tau}_6 \cdot \bar{\mathcal{R}}^{\mathbb{O}\oplus\mathbb{Z}_2^c} = \{0\}$ as well. The inclusion " $\supset$ " in (4.3) is obvious. To show equality use that the generators of $\bar{\mathcal{R}}^{\mathbb{T}}$ are $\bar{\tau}_3, \bar{\rho}_4$ and $\bar{\tau}_6$ by Corollary 4.4. An arbitrary polynomial $\bar{q} \in \bar{\mathcal{R}}^{\mathbb{T}}$ is therefore of the form

$$\bar{q} = \sum \alpha_{i,j,m} \bar{\tau}_3^i \bar{\rho}_4^j \bar{\tau}_6^m = \sum_{i \,\text{even}} \alpha_{i,j,m} \bar{\tau}_3^i \bar{\rho}_4^j \bar{\tau}_6^m + \bar{\tau}_3 \sum_{i \,\text{odd}} \alpha_{i,j,m} \bar{\tau}_3^{i-1} \bar{\rho}_4^j \bar{\tau}_6^m$$

and (4.3) is established, since $\bar{\tau}_3^2 \in \bar{\mathcal{R}}^{\mathbb{T}\oplus\mathbb{Z}_2^c}$ and $\bar{\rho}_4$ as well as $\bar{\tau}_6$ are generators of $\bar{\mathcal{R}}^{\mathbb{T}\oplus\mathbb{Z}_2^c}$. These two together with $\bar{\rho}_6$ are all generators of $\bar{\mathcal{R}}^{\mathbb{T}\oplus\mathbb{Z}_2^c}$. Hence, an arbitrary polynomial $\bar{q} \in \bar{\mathcal{R}}^{\mathbb{T}\oplus\mathbb{Z}_2^c}$ is of the form

$$\bar{q} = \sum \beta_{i,j,m} \bar{\rho}_4^i \bar{\tau}_6^j \bar{\rho}_6^m.$$

We can argue as above, since $\bar{\tau}_6^2 \in \bar{\mathcal{R}}^{\mathbb{O}\oplus\mathbb{Z}_2^c}$ and the generators of $\bar{\mathcal{R}}^{\mathbb{O}\oplus\mathbb{Z}_2^c}$ are $\bar{\rho}_4$ and $\bar{\rho}_6$. Again " $\supset$ " it trivial and the theorem is proved. □

Observe that $\bar{\tau}_3 \in \bar{\mathcal{R}}^{\mathbb{O}^-}$ gives

$$\bar{\tau}_3 \cdot \bar{\mathcal{R}}^{\mathbb{O}\oplus\mathbb{Z}_2^c} \subset \bar{\mathcal{R}}^{\mathbb{O}^-}, \text{ whereas } \bar{\tau}_3\bar{\tau}_6 \cdot \bar{\mathcal{R}}^{\mathbb{O}\oplus\mathbb{Z}_2^c} \subset \bar{\mathcal{R}}^{\mathbb{O}}$$

follows from $\bar{\tau}_3\bar{\tau}_6 \in \bar{\mathcal{R}}^{\mathbb{O}}$. Actually we even have:

Theorem 4.7 *Let $\mathbb{T} \subset \mathbf{O}(3)$ be fixed as in Subsection 3.1.1 and let $\mathbb{O}^-$ and $\mathbb{O}\oplus\mathbb{Z}_2^c$ be supergroups of $\mathbb{T}$. Then*

$$\begin{aligned} \bar{\mathcal{R}}^{\mathbb{O}^-} &= \bar{\mathcal{R}}^{\mathbb{O}\oplus\mathbb{Z}_2^c} \oplus \bar{\tau}_3 \cdot \bar{\mathcal{R}}^{\mathbb{O}\oplus\mathbb{Z}_2^c} && (4.6)\\ \text{and } \bar{\mathcal{R}}^{\mathbb{O}} &= \bar{\mathcal{R}}^{\mathbb{O}\oplus\mathbb{Z}_2^c} \oplus \bar{\tau}_3\bar{\tau}_6 \cdot \bar{\mathcal{R}}^{\mathbb{O}\oplus\mathbb{Z}_2^c} && (4.7) \end{aligned}$$

holds, where again both decompositions are orthogonal in $L^2(S^2)$.

Proof. Let $\bar{q}$ be an arbitrary polynomial in $\bar{\mathcal{R}}^{\mathbb{O}\oplus\mathbb{Z}_2^c}$. Then $\bar{q} \in \bar{\mathcal{R}}^{\mathbb{T}\oplus\mathbb{Z}_2^c}$ as well as $\bar{\tau}_6\bar{q} \in \bar{\mathcal{R}}^{\mathbb{T}\oplus\mathbb{Z}_2^c}$. Consequently, Theorem 4.6 provides

$$(\bar{\tau}_3, \bar{q})_{L^2(S^2)} = 0 \text{ and } (\bar{\tau}_3\bar{\tau}_6, \bar{q})_{L^2(S^2)} = 0.$$

It remains to show "$\supset$" in (4.6) and (4.7). The generators of $\bar{\mathcal{R}}^{\mathbb{O}^-}$ are $\bar{\tau}_3$ and $\bar{\rho}_4$. Therefore an arbitrary polynomial $\bar{q} \in \bar{\mathcal{R}}^{\mathbb{O}^-}$ is of the form

$$\bar{q} = \sum \alpha_{i,j} \bar{\tau}_3^i \bar{\rho}_4^j.$$

Again $\bar{\tau}_3^2 \in \bar{\mathcal{R}}^{\mathbb{O}\oplus\mathbb{Z}_2^c}$ yields the missing argument, if we proceed as in the proof of Theorem 4.6. (4.7) is proved in the same way, using the generators of $\bar{\mathcal{R}}^{\mathbb{O}}$ ($\bar{\rho}_4, \bar{\rho}_6$ and $\bar{\tau}_3\bar{\tau}_6$) and the fact that $(\bar{\tau}_3\bar{\tau}_6)^2 \in \bar{\mathcal{R}}^{\mathbb{O}\oplus\mathbb{Z}_2^c}$. □

From the above theorem we conclude, that the first three components of the decomposition (4.5) have actually more symmetry than only $\mathbb{T}$ or $\mathbb{T}\oplus\mathbb{Z}_2^c$. Moreover, the elements in $\bar{\tau}_3\bar{\mathcal{R}}^{\mathbb{O}\oplus\mathbb{Z}_2^c}$ are the elements with exactly $\mathbb{O}^-$ symmetry (and not more!), whereas the ones in $\bar{\tau}_3\bar{\tau}_6\bar{\mathcal{R}}^{\mathbb{O}\oplus\mathbb{Z}_2^c}$ have exactly $\mathbb{O}$ symmetry.

For $U^{\bar{\mathcal{R}}} = \bar{\tau}_6\bar{\mathcal{R}}^{\mathbb{O}\oplus\mathbb{Z}_2^c}$ we observe $U^{\bar{\mathcal{R}}} \subset \bar{\mathcal{R}}^{\mathbb{T}\oplus\mathbb{Z}_2^c}$, but some elements in $U^{\bar{\mathcal{R}}}$ have in some sense even more symmetry: let $\mathbb{I}$ be the supergroup of $\mathbb{T}$ introduced in Section 3 and $V^{\bar{\mathcal{R}}} := \mathrm{Proj}_{U^{\bar{\mathcal{R}}}}(\bar{\mathcal{R}}^{\mathbb{I}\oplus\mathbb{Z}_2^c}) \subset U^{\bar{\mathcal{R}}}$ (here by $\mathrm{Proj}_{U^{\bar{\mathcal{R}}}}$ we mean the orthogonal projection on $U^{\bar{\mathcal{R}}}$ resulting from the decomposition (4.4)). The space $U^{\bar{\mathcal{R}}}$ decomposes orthogonally to

$$U^{\bar{\mathcal{R}}} = V^{\bar{\mathcal{R}}} \oplus W^{\bar{\mathcal{R}}}, \tag{4.8}$$

where $W^{\bar{\mathcal{R}}} := \{\bar{u} \in U^{\bar{\mathcal{R}}} \mid (\bar{u},\bar{v})_{L^2(S^2)} = 0,\ \forall \bar{v} \in V^{\bar{\mathcal{R}}}\} = \mathrm{Proj}^{\perp}_{U^{\bar{\mathcal{R}}}}(\bar{\mathcal{R}}^{\mathbb{I}\oplus\mathbb{Z}_2^c}) \subset U^{\bar{\mathcal{R}}}$. We claim

$$V^{\bar{\mathcal{R}}} \subset \mathrm{Span}\{\bar{\mathcal{R}}^{\mathbb{O}\oplus\mathbb{Z}_2^c}, \bar{\mathcal{R}}^{\mathbb{I}\oplus\mathbb{Z}_2^c}\}. \tag{4.9}$$

To see that note $V^{\bar{\mathcal{R}}} \subset U^{\bar{\mathcal{R}}} = \bar{\tau}_6\bar{\mathcal{R}}^{\mathbb{O}\oplus\mathbb{Z}_2^c}$ and (4.4) implies that an arbitrary $\bar{q} \in \bar{\mathcal{R}}^{\mathbb{I}\oplus\mathbb{Z}_2^c} \subset \bar{\mathcal{R}}^{\mathbb{T}\oplus\mathbb{Z}_2^c}$ can be written as $\bar{q} = \bar{q}_1 + \bar{\tau}_6\bar{q}_2$ with both $\bar{q}_1$ and $\bar{q}_2$ in $\bar{\mathcal{R}}^{\mathbb{O}\oplus\mathbb{Z}_2^c}$. This gives

$$V^{\bar{\mathcal{R}}} \ni \bar{v} := \mathrm{Proj}_{U^{\bar{\mathcal{R}}}}(\bar{q}) = \bar{\tau}_6\bar{q}_2 = \bar{q} - \bar{q}_1$$

and (4.9) follows. Hence, the elements in $V^{\bar{\mathcal{R}}}$ can all be written as a sum of two polynomials with the additional symmetry $\mathbb{O}\oplus\mathbb{Z}_2^c$ or $\mathbb{I}\oplus\mathbb{Z}_2^c$, respectively. Only the space $W^{\bar{\mathcal{R}}}$ seems to have pure $\mathbb{T}\oplus\mathbb{Z}_2^c$ symmetry:

Theorem 4.8 *Let again $\mathbb{T} \subset \mathbf{O}(3)$ be as in Section 3 and let $\mathbb{O}$ as well as $\mathbb{I}$ be supergroups of $\mathbb{T}$ as before. Using the spaces $V^{\bar{\mathcal{R}}}$ and $W^{\bar{\mathcal{R}}}$ defined above we claim*

$$\bar{\mathcal{R}}^{\mathbb{O}\oplus\mathbb{Z}_2^c} \oplus V^{\bar{\mathcal{R}}} = \mathrm{Span}\{\bar{\mathcal{R}}^{\mathbb{O}\oplus\mathbb{Z}_2^c}, \bar{\mathcal{R}}^{\mathbb{I}\oplus\mathbb{Z}_2^c}\}. \tag{4.10}$$

Consequently,

$$\begin{aligned} W^{\bar{\mathcal{R}}} &\perp \mathrm{Span}\{\bar{\mathcal{R}}^{\mathbb{O}\oplus\mathbb{Z}_2^c}, \bar{\mathcal{R}}^{\mathbb{I}\oplus\mathbb{Z}_2^c}\} && (4.11)\\ \textit{and } \bar{\mathcal{R}}^{\mathbb{T}\oplus\mathbb{Z}_2^c} &= \mathrm{Span}\{\bar{\mathcal{R}}^{\mathbb{O}\oplus\mathbb{Z}_2^c}, \bar{\mathcal{R}}^{\mathbb{I}\oplus\mathbb{Z}_2^c}\} \oplus W^{\bar{\mathcal{R}}} && (4.12)\end{aligned}$$

holds. Furthermore, $W^{\bar{\mathcal{R}}}$ is independent of the particular choice of $\mathbb{I} \supset \mathbb{T}$ (cf. Subsubsection 3.1.6).

Proof. We begin with (4.10). $\bar{\mathcal{R}}^{\mathbb{O}\oplus\mathbb{Z}_2^c} + V^{\bar{\mathcal{R}}} = \mathrm{Span}\{\bar{\mathcal{R}}^{\mathbb{O}\oplus\mathbb{Z}_2^c}, \bar{\mathcal{R}}^{\mathbb{I}\oplus\mathbb{Z}_2^c}\}$ is obvious from (4.9) and the sum is direct, because it is even orthogonal due to $V^{\bar{\mathcal{R}}} \subset U^{\bar{\mathcal{R}}} \perp \bar{\mathcal{R}}^{\mathbb{O}\oplus\mathbb{Z}_2^c}$. Both (4.11) and (4.12) follow immediately from (4.4) and (4.8).

It remains to show the independence of $W^{\bar{\mathcal{R}}}$ of the particular choice of $\mathbb{I} \supset \mathbb{T}$. Suppose $\tilde{\mathbb{I}} \supset \mathbb{T}$ is the other copy of a icosahedral supergroup of $\mathbb{T}$ as in Subsubsection 3.1.6 introduced. We claim

$$\mathrm{Proj}_{U^{\bar{\mathcal{R}}}}(\bar{\mathcal{R}}^{\mathbb{I}\oplus\mathbb{Z}_2^c}) = \mathrm{Proj}_{U^{\bar{\mathcal{R}}}}(\bar{\mathcal{R}}^{\tilde{\mathbb{I}}\oplus\mathbb{Z}_2^c}).$$

To see this let $\xi_4 \in \mathbb{O}\backslash\mathbb{T}$ be an element of order 4 in $\mathbb{O} \supset \mathbb{T}$. As already seen in Subsubsection 3.1.6, ξ_4 conjugates $\mathbb{I}$ to $\tilde{\mathbb{I}}$: $\xi_4^{-1}\mathbb{I}\xi_4 = \tilde{\mathbb{I}}$. Therefore with $\bar{p} \in \bar{\mathcal{R}}^{\mathbb{I}\oplus\mathbb{Z}_2^c}$ we have $\bar{q} := \xi_4\bar{p} \in \bar{\mathcal{R}}^{\tilde{\mathbb{I}}\oplus\mathbb{Z}_2^c}$. Writing $\bar{p} = \mathrm{Proj}_{\bar{\mathcal{R}}^{\mathbb{O}\oplus\mathbb{Z}_2^c}}(\bar{p}) + \mathrm{Proj}_{U^{\bar{\mathcal{R}}}}(\bar{p})$ we infer

$$\bar{q} = \xi_4\bar{p} = \mathrm{Proj}_{\bar{\mathcal{R}}^{\mathbb{O}\oplus\mathbb{Z}_2^c}}(\bar{p}) - \mathrm{Proj}_{U^{\bar{\mathcal{R}}}}(\bar{p}),$$

since the action of ξ_4 on elements of $\bar{\mathcal{R}}^{\mathbb{O}\oplus\mathbb{Z}_2^c}$ is trivial and elements in $U^{\bar{\mathcal{R}}}$ obtain a minus. Therefore the projection of $\bar{\mathcal{R}}^{\mathbb{I}\oplus\mathbb{Z}_2^c}$ and $\bar{\mathcal{R}}^{\tilde{\mathbb{I}}\oplus\mathbb{Z}_2^c}$ to $U^{\bar{\mathcal{R}}}$ span the same space. □

The elements in $W^{\bar{\mathcal{R}}}$ will be of major interest to us, since they contain all elements with precise $\mathbb{T}\oplus\mathbb{Z}_2^c$ symmetry. Still it is by no means clear how large $W^{\bar{\mathcal{R}}}$ is and how we can calculate the elements of $W^{\bar{\mathcal{R}}}$. The following definition provides subspaces, which eventually give the decomposition of $W^{\bar{\mathcal{R}}}$. For the rest of this section we are only interested in polynomials with at least $\mathbb{T}\oplus\mathbb{Z}_2^c$ symmetry. Observe that the elements of $\bar{\mathcal{R}}^{\mathbb{T}\oplus\mathbb{Z}_2^c}$ are all restrictions of polynomials of even degree (cf. Corollary 4.4), so we do not have to worry about any odd degree polynomials.

Definition 4.9 *Let $W_{2j}^{\bar{\mathcal{R}}}$, $j \geq 0$ be recursively defined as the maximal subspace of $\bar{\mathcal{R}}_{2j}^{\mathbb{T}\oplus\mathbb{Z}_2^c} \cap U^{\bar{\mathcal{R}}} = \bar{\mathcal{R}}_{\leq 2j}^{\mathbb{T}\oplus\mathbb{Z}_2^c} \cap U^{\bar{\mathcal{R}}} \subset U^{\bar{\mathcal{R}}}$ which satisfies the condition*

$$W_{2j}^{\bar{\mathcal{R}}} \perp \mathrm{Span}\{\bar{\mathcal{R}}_{\leq 2j}^{\mathbb{I}\oplus\mathbb{Z}_2^c}, \bigoplus_{i=0}^{j-1} W_{2i}^{\bar{\mathcal{R}}}\}. \tag{4.13}$$

Some of these subspaces will only contain 0, and therefore these subspaces won't contribute much to our decomposition. Theorem 4.11 will tell us exactly which of them. We have:

Theorem 4.10 *$(W_{2j}^{\bar{\mathcal{R}}})_{j\geq 0}$ is a sequence of pairwise orthogonal subspaces in $L^2(S^2)$ which satisfy*

$$W_{2j}^{\bar{\mathcal{R}}} \perp \mathrm{Span}\{\bar{\mathcal{R}}^{\mathbb{O}\oplus\mathbb{Z}_2^c}, \bar{\mathcal{R}}^{\mathbb{I}\oplus\mathbb{Z}_2^c}\}. \tag{4.14}$$

In particular, they form an orthogonal decomposition of $W^{\bar{\mathcal{R}}}$:

$$W^{\bar{\mathcal{R}}} = \bigoplus_{j=0}^{\infty} W_{2j}^{\bar{\mathcal{R}}}. \tag{4.15}$$

Proof. To start with (4.14) first of all note that $W_{2j}^{\bar{\mathcal{R}}} \subset U^{\bar{\mathcal{R}}} \perp \bar{\mathcal{R}}^{\mathbb{O}\oplus\mathbb{Z}_2^c}$. Suppose

$$\bar{w} \in W_{2j}^{\bar{\mathcal{R}}} \perp \bar{\mathcal{R}}_{\leq 2j}^{\mathbb{I}\oplus\mathbb{Z}_2^c} \tag{4.16}$$

is given. We have to prove $\bar{w} \perp \bar{\mathcal{R}}^{\mathbb{I}\oplus\mathbb{Z}_2^c}$. Our proof uses projections on fixed-point spaces: for $\bar{p} \in \bar{\mathcal{R}}$ define

$$Q_{\bar{\mathcal{R}}}^{L}(\bar{p}) := \mathrm{Proj}_{\bar{\mathcal{R}}^L}(\bar{p}) = \frac{1}{|L|}\sum_{\gamma\in L}\gamma\bar{p} \in \bar{\mathcal{R}}^{L}.$$

Now if $\bar{q} \in \bar{\mathcal{R}}^{\mathbb{I}\oplus\mathbb{Z}_2^c}$ in order to show $(\bar{w},\bar{q})_{L^2(S^2)} = 0$, it is sufficient to show $Q_{\bar{\mathcal{R}}}^{\mathbb{Z}_5}(\bar{w}) = 0$ for some $\mathbb{Z}_5 \subset \mathbb{I}$. To see that let $\xi_5 \in \mathbb{Z}_5$ be one of its generators. Then with $\xi_5^5 = \mathbb{1}$ we find

$$\begin{aligned}(\bar{w},\bar{q})_{L^2(S^2)} &= (\bar{w}, Q_{\bar{\mathcal{R}}}^{\mathbb{Z}_5}(\bar{q}))_{L^2(S^2)} = \frac{1}{5}\sum_{i=0}^{4}(\bar{w},\xi_5^i\bar{q})_{L^2(S^2)} \\ &= \frac{1}{5}\sum_{i=0}^{4}(\xi_5^{5-i}\bar{w},\xi_5^5\bar{q})_{L^2(S^2)} = (Q_{\bar{\mathcal{R}}}^{\mathbb{Z}_5}(\bar{w}),\bar{q})_{L^2(S^2)}.\end{aligned}$$

Now obviously $\tilde{w} := Q_{\bar{\mathcal{R}}}^{\mathbb{Z}_5}(\bar{w}) \in \bar{\mathcal{R}}_{\leq 2j}^{\mathbb{Z}_5}$. On the other hand we will show in a moment that $\tilde{w} \in (\bar{\mathcal{R}}_{\leq 2j}^{\mathbb{Z}_5})^{\perp} \subset \bar{\mathcal{R}}_{\leq 2j}$, which is only possible if $\tilde{w} = 0$ and the proof would be accomplished. The remaining: for $\bar{p} \in \bar{\mathcal{R}}_{\leq 2j}^{\mathbb{Z}_5}$ we have

$$\begin{aligned}(\tilde{w},\bar{p})_{L^2(S^2)} &= (Q_{\bar{\mathcal{R}}}^{\mathbb{Z}_5}(\bar{w}),\bar{p})_{L^2(S^2)} = (\bar{w}, Q_{\bar{\mathcal{R}}}^{\mathbb{Z}_5}(\bar{p}))_{L^2(S^2)} = (\bar{w},\bar{p})_{L^2(S^2)} \\ &= (Q_{\bar{\mathcal{R}}}^{\mathbb{T}\oplus\mathbb{Z}_2^c}(\bar{w}), Q_{\bar{\mathcal{R}}}^{\mathbb{T}\oplus\mathbb{Z}_2^c}(\bar{p}))_{L^2(S^2)} = (\bar{w}, Q_{\bar{\mathcal{R}}}^{\mathbb{T}\oplus\mathbb{Z}_2^c}(\bar{p}))_{L^2(S^2)}\end{aligned}$$

since $\bar{w} \in \bar{\mathcal{R}}^{\mathbb{T}\oplus\mathbb{Z}_2^c}$. Using again that $\bar{p} \in \bar{\mathcal{R}}_{\leq 2j}^{\mathbb{Z}_5}$ and (4.16), we conclude

$$\begin{aligned}(\tilde{w},\bar{p})_{L^2(S^2)} &= (\bar{w}, \frac{1}{24}\sum_{\gamma\in\mathbb{T}\oplus\mathbb{Z}_2^c}\gamma\bar{p})_{L^2(S^2)} \\ &= (\bar{w}, \frac{1}{24}\sum_{\gamma\in\mathbb{T}\oplus\mathbb{Z}_2^c}\frac{1}{5}\sum_{i=0}^{4}\gamma\xi_5^i\bar{p})_{L^2(S^2)} = (\bar{w}, \frac{1}{120}\sum_{\gamma\in\mathbb{I}\oplus\mathbb{Z}_2^c}\gamma\bar{p})_{L^2(S^2)} = 0,\end{aligned}$$

since $\sum_{\gamma\in\mathbb{I}\oplus\mathbb{Z}_2^c}\gamma\bar{p}\in\bar{\mathcal{R}}_{\leq 2j}^{\mathbb{I}\oplus\mathbb{Z}_2^c}$. It remains to prove (4.15). "$\supset$" follows immediately from (4.14) and the definition of $W^{\bar{\mathcal{R}}}$. To see "$\subset$", let $\bar{w}\in W^{\bar{\mathcal{R}}}\subset U^{\bar{\mathcal{R}}}$ be given. Then $\bar{w}\in U^{\bar{\mathcal{R}}}=\bar{\tau}_6\bar{\mathcal{R}}^{\mathbb{O}\oplus\mathbb{Z}_2^c}\subset\bar{\mathcal{R}}^{\mathbb{T}\oplus\mathbb{Z}_2^c}$. Since $\bar{w}$ must be a restriction of a polynomial of finite degree, we conclude even $\bar{w}\in\bar{\mathcal{R}}_{\leq 2j}^{\mathbb{T}\oplus\mathbb{Z}_2^c}$ for some $j\geq 0$. But since $\bar{w}\perp\mathrm{Span}\{\bar{\mathcal{R}}^{\mathbb{O}\oplus\mathbb{Z}_2^c},\bar{\mathcal{R}}^{\mathbb{I}\oplus\mathbb{Z}_2^c}\}$, certainly also $\bar{w}\perp\bar{\mathcal{R}}_{\leq 2j}^{\mathbb{I}\oplus\mathbb{Z}_2^c}$ holds and by Definition 4.9 $\bar{w}\in\bigoplus_{i=0}^{j}W_{2i}^{\bar{\mathcal{R}}}$ follows, which proves everything. □

The last theorem in this section will tell us how large $W_{2j}^{\bar{\mathcal{R}}}$ actually is.

Theorem 4.11 *We obtain for any $j\geq 0$*

$$\bar{\mathcal{R}}_{\leq 2j}^{\mathbb{T}\oplus\mathbb{Z}_2^c}=\mathrm{Span}\left\{\bar{\mathcal{R}}_{\leq 2j}^{\mathbb{O}\oplus\mathbb{Z}_2^c},\bar{\mathcal{R}}_{\leq 2j}^{\mathbb{I}\oplus\mathbb{Z}_2^c}\right\}\oplus\bigoplus_{i=0}^{j}W_{2i}^{\bar{\mathcal{R}}}. \tag{4.17}$$

Furthermore, the dimension of $W_{2j}^{\bar{\mathcal{R}}}$ is given by the coefficient of s^{2j} in the Poincaré–series

$$\begin{aligned} P_{\bar{\mathcal{R}}}^{W}(s) &:= \frac{s^{14}}{(1-s^4)(1-s^{10})} \qquad (4.18)\\ &= s^{14}+s^{18}+s^{22}+s^{24}+s^{26}+s^{28}+s^{30}+s^{32}+2\,s^{34}+s^{36}+2\,s^{38}+O\left(s^{40}\right). \end{aligned}$$

Proof. Equation (4.17) follows immediately from (4.12) and (4.15) by projecting both sides to $\bar{\mathcal{R}}_{\leq 2j}^{\mathbb{T}\oplus\mathbb{Z}_2^c}$.

The space $\bar{\mathcal{R}}_{\leq 2j}^{\mathbb{T}\oplus\mathbb{Z}_2^c}$ decomposes by Proposition 4.5 to $\bigoplus_{i=0}^{j}\bar{\mathcal{Q}}_{2i}^{\mathbb{T}\oplus\mathbb{Z}_2^c}$ and similarly

$$\bar{\mathcal{R}}_{\leq 2j}^{\mathbb{O}\oplus\mathbb{Z}_2^c}=\bigoplus_{i=0}^{j}\bar{\mathcal{Q}}_{2i}^{\mathbb{O}\oplus\mathbb{Z}_2^c}\text{ and }\bar{\mathcal{R}}_{\leq 2j}^{\mathbb{I}\oplus\mathbb{Z}_2^c}=\bigoplus_{i=0}^{j}\bar{\mathcal{Q}}_{2i}^{\mathbb{I}\oplus\mathbb{Z}_2^c}.$$

The sum $\bar{\mathcal{R}}_{\leq 2j}^{\mathbb{O}\oplus\mathbb{Z}_2^c}+\bar{\mathcal{R}}_{\leq 2j}^{\mathbb{I}\oplus\mathbb{Z}_2^c}$ is not direct, but $\bar{\mathcal{R}}_{\leq 2j}^{\mathbb{O}\oplus\mathbb{Z}_2^c}\cap\bar{\mathcal{R}}_{\leq 2j}^{\mathbb{I}\oplus\mathbb{Z}_2^c}$ contains only constants: every polynomial having both $\mathbb{O}\oplus\mathbb{Z}_2^c$ and $\mathbb{I}\oplus\mathbb{Z}_2^c$ symmetry must have already $\mathbf{O}(3)$ symmetry, since both subgroups are maximal. The Poincaré–series of $\mathbf{O}(3)$ is $P_{\mathcal{R}}^{\mathbf{O}(3)}(s)=\frac{1}{1-s^2}$ and ρ_2 is the only generator of $\mathcal{R}^{\mathbf{O}(3)}$. Hence $\bar{\mathcal{R}}^{\mathbf{O}(3)}=\mathbb{R}[1]$ is one dimensional. We therefore find for $j\geq 0$:

$$\begin{aligned} \dim W_{2j}^{\bar{\mathcal{R}}} &= \dim\bar{\mathcal{Q}}_{2j}^{\mathbb{T}\oplus\mathbb{Z}_2^c}-\dim\,\mathrm{Span}\left\{\bar{\mathcal{Q}}_{2j}^{\mathbb{O}\oplus\mathbb{Z}_2^c},\bar{\mathcal{Q}}_{2j}^{\mathbb{I}\oplus\mathbb{Z}_2^c}\right\}\\ &= \dim\bar{\mathcal{Q}}_{2j}^{\mathbb{T}\oplus\mathbb{Z}_2^c}-(\dim\bar{\mathcal{Q}}_{2j}^{\mathbb{O}\oplus\mathbb{Z}_2^c}+\dim\bar{\mathcal{Q}}_{2j}^{\mathbb{I}\oplus\mathbb{Z}_2^c}-\dim\bar{\mathcal{Q}}_{2j}^{\mathbf{O}(3)}), \end{aligned}$$

which is by Proposition 4.5 given by the $2j$–th coefficient of the Poincaré–series

$$
\begin{aligned}
P_{\bar{\mathcal{R}}}^{W}(s) &= P_{\bar{\mathcal{R}}}^{\mathbb{T}\oplus\mathbb{Z}_2^c}(s) - (P_{\bar{\mathcal{R}}}^{\mathbb{O}\oplus\mathbb{Z}_2^c}(s) + P_{\bar{\mathcal{R}}}^{\mathbb{I}\oplus\mathbb{Z}_2^c}(s) - P_{\bar{\mathcal{R}}}^{\mathbf{O}(3)}(s)) \\
&= \frac{1+s^6}{(1-s^4)(1-s^6)} - \left(\frac{1}{(1-s^4)(1-s^6)} + \frac{1}{(1-s^6)(1-s^{10})} - 1\right) \\
&= \frac{s^6}{(1-s^4)(1-s^6)} + \frac{-s^6-s^{10}+s^{16}}{(1-s^6)(1-s^{10})} \\
&= \frac{s^6(1-s^{10})}{(1-s^4)(1-s^6)(1-s^{10})} + \frac{-s^6+s^{14}+s^{16}-s^{20}}{(1-s^4)(1-s^6)(1-s^{10})} \\
&= \frac{s^{14}-s^{20}}{(1-s^4)(1-s^6)(1-s^{10})} = \frac{s^{14}}{(1-s^4)(1-s^{10})}.
\end{aligned}
$$

□

Note that although $W^{\bar{\mathcal{R}}}$ has a Poincaré–series, $W^{\bar{\mathcal{R}}}$ is by no means an algebra! The somehow cumbersome definition of $W_{2j}^{\bar{\mathcal{R}}}$ turns now out to be very helpful for calculating basises of these spaces. We have e.g. $W_{14}^{\bar{\mathcal{R}}} = \mathrm{Span}\{\bar{w}_{14}^{\bar{\mathcal{R}}}\}$ and $W_{18}^{\bar{\mathcal{R}}} = \mathrm{Span}\{\bar{w}_{18}^{\bar{\mathcal{R}}}\}$ with

$$
\bar{w}_{14}^{\bar{\mathcal{R}}} := \bar{\tau}_6 \cdot \left(\frac{23}{135}\bar{\rho}_2^4 - \frac{22}{45}\bar{\rho}_2^2\bar{\rho}_4 - \frac{16}{27}\bar{\rho}_2\bar{\rho}_6 + \bar{\rho}_4^2\right) \tag{4.19}
$$

$$
\begin{aligned}
\bar{w}_{18}^{\bar{\mathcal{R}}} := \bar{\tau}_6 \cdot \Big(&\frac{8893}{4455}\bar{\rho}_2^6 + \frac{1837}{135}\bar{\rho}_2^4\bar{\rho}_4 - \frac{42544}{4455}\bar{\rho}_2^3\bar{\rho}_6 - \frac{2347}{99}\bar{\rho}_2^2\bar{\rho}_4^2 \\
&+\frac{4496}{135}\bar{\rho}_2\bar{\rho}_4\bar{\rho}_6 + \bar{\rho}_4^3 - \frac{1024}{81}\bar{\rho}_6^2\Big),
\end{aligned} \tag{4.20}
$$

where one only has to check $\bar{w}_{14}^{\bar{\mathcal{R}}} \perp \{\bar{\iota}_6, \bar{\iota}_{10}, \bar{\iota}_6^2\}$ and $\bar{w}_{18}^{\bar{\mathcal{R}}} \perp \{\bar{\iota}_6, \bar{\iota}_{10}, \bar{\iota}_6^2, \bar{\iota}_6\bar{\iota}_{10}, \bar{\iota}_6^3, \bar{w}_{14}^{\bar{\mathcal{R}}}\}$. This is left to the reader.

4.2 Orthogonal Decomposition of $\bar{\mathcal{M}}^{\mathbb{T}}$

Our next goal is to answer the question on the precise tetrahedral symmetry for the equivariants as well. Our proceeding will be very similar to the one in the preceding subsection. Particularly, we will skip arguments whenever things work out the same way. A mapping $b : \mathbb{R}^3 \to \mathbb{R}^3$ is equivariant with respect to a subgroup $L \subset \mathbf{O}(3)$ if

$$
\gamma b(\zeta) = b(\gamma\zeta), \text{ for all } \zeta \in \mathbb{R}^3 \text{ and } \gamma \in L.
$$

The related $L-$action on mappings from $\mathbb{R}^3$ into $\mathbb{R}^3$ is defined by

$$
(\gamma b)(\zeta) := \gamma b(\gamma^{-1}\zeta), \quad \zeta \in \mathbb{R}^3 \text{ and } \gamma \in L \subset \mathbf{O}(3). \tag{4.21}
$$

Obviously $b : \mathbb{R}^3 \to \mathbb{R}^3$ is $L-$equivariant if and only if b is invariant with respect to this $L-$action (i.e. $\gamma b = b$ for all $\gamma \in L$). We start defining for the equivariants similar linear spaces as we did for the invariants.

Definition 4.12

$$\begin{aligned}
\mathcal{M} &:= \{e : \mathbb{R}^3 \to \mathbb{R}^3 \mid e \textit{ is polynomial } \} \\
\mathcal{M}_i &:= \{e \in \mathcal{M} \mid e \textit{ is homogeneous and } \deg(e) = i\} \\
\mathcal{M}_{\leq i} &:= \bigoplus_{j=0}^{i} \mathcal{M}_j = \{e \in \mathcal{M} \mid \deg(e) \leq i\} \\
\mathcal{M}^L &:= \{e \in \mathcal{M} \mid \gamma e = e \textit{ for all } \gamma \in L\}.
\end{aligned}$$

The spaces $\mathcal{M}_i^L$ and $\mathcal{M}_{\leq i}^L$ are defined analogously.

Resuming the results of the last section on equivariants we know a minimal set of generators for the modules $\mathcal{M}^L$:

Corollary 4.13

$$\begin{aligned}
\mathcal{M}^{\mathbb{T}} &= < \epsilon_1, \epsilon_2, \epsilon_{3a}, \epsilon_{3b}, \epsilon_4, \epsilon_5 >_{\mathcal{R}^{\mathbb{O}^-}}, \quad \mathcal{M}^{\mathbb{T}\oplus\mathbb{Z}_2^c} =< \epsilon_1, \epsilon_{3a}, \epsilon_{3b}, \epsilon_5, \tau_3\epsilon_2, \tau_3\epsilon_4 >_{\mathcal{R}^{\mathbb{O}\oplus\mathbb{Z}_2^c}} \\
\mathcal{M}^{\mathbb{O}} &= < \epsilon_1, \epsilon_{3a}, \epsilon_4, \tau_3\epsilon_2, \tau_3\epsilon_{3b}, \tau_3\epsilon_5 >_{\mathcal{R}^{\mathbb{O}\oplus\mathbb{Z}_2^c}}, \quad \mathcal{M}^{\mathbb{O}^-} =< \epsilon_1, \epsilon_2, \epsilon_{3a} >_{\mathcal{R}^{\mathbb{O}^-}} \\
\mathcal{M}^{\mathbb{O}\oplus\mathbb{Z}_2^c} &= < \epsilon_1, \epsilon_{3a}, \tau_3\epsilon_2 >_{\mathcal{R}^{\mathbb{O}\oplus\mathbb{Z}_2^c}}, \quad \mathcal{M}^{\mathbb{I}\oplus\mathbb{Z}_2^c} =< \epsilon_1, \nabla\iota_6, \nabla\iota_{10} >_{\mathcal{R}^{\mathbb{I}\oplus\mathbb{Z}_2^c}} .
\end{aligned}$$

The dimension of $\mathcal{M}_i^L, L = \mathbb{T}, \mathbb{T}\oplus\mathbb{Z}_2^c, \mathbb{O}, \mathbb{O}^-, \mathbb{O}\oplus\mathbb{Z}_2^c, \mathbb{I}$ and $\mathbb{I}\oplus\mathbb{Z}_2^c$, is given by the i-th coefficient of the Poincaré–series $P_{\mathcal{M}}^L$ (cf. Section 3).

Proof. The last three statements are obvious from our previous results. In the first three statements one containment relation is also obvious. The other one is obtained from the Poincaré–series. There is a Poincaré–series associated to the module generated by elements on the right hand side over the respective ring. It can be easily checked that it coincides with the Poincaré–series for the left hand side. By inclusion the two sides are equal. $\square$

Actually, we are again only interested in the restrictions of the above polynomial mappings to the sphere S^2. Therefore let

Definition 4.14

$$\bar{\mathcal{M}} := \{\bar{e} : S^2 \to \mathbb{R}^3 \mid \exists e \in \mathcal{M} \textit{ with } e_{|S^2} = \bar{e}\}$$

and similarly, define $\bar{\mathcal{M}}_i, \bar{\mathcal{M}}_{\leq i}, \bar{\mathcal{M}}^L, \bar{\mathcal{M}}_i^L$ and $\bar{\mathcal{M}}_{\leq i}^L$ as linear spaces of the restrictions of the appropriate polynomial mappings.

Again it follows immediately:

Corollary 4.15

$$\begin{aligned}
\bar{\mathcal{M}}^{\mathbb{T}} &= <\bar\epsilon_1,\bar\epsilon_2,\bar\epsilon_{3a},\bar\epsilon_{3b},\bar\epsilon_4,\bar\epsilon_5>_{\bar{\mathcal{R}}^{\mathbb{O}^-}}, \quad \bar{\mathcal{M}}^{\mathbb{T}\oplus\mathbb{Z}_2^c} =<\bar\epsilon_1,\bar\epsilon_{3a},\bar\epsilon_{3b},\bar\epsilon_5,\bar\tau_3\bar\epsilon_2,\bar\tau_3\bar\epsilon_4>_{\bar{\mathcal{R}}^{\mathbb{O}\oplus\mathbb{Z}_2^c}}\\
\bar{\mathcal{M}}^{\mathbb{O}} &= <\bar\epsilon_1,\bar\epsilon_{3a},\bar\epsilon_4,\bar\tau_3\bar\epsilon_2,\bar\tau_3\bar\epsilon_{3b},\bar\tau_3\bar\epsilon_5>_{\bar{\mathcal{R}}^{\mathbb{O}\oplus\mathbb{Z}_2^c}}, \quad \bar{\mathcal{M}}^{\mathbb{O}^-} =<\bar\epsilon_1,\bar\epsilon_2,\bar\epsilon_{3a}>_{\bar{\mathcal{R}}^{\mathbb{O}^-}}\\
\bar{\mathcal{M}}^{\mathbb{O}\oplus\mathbb{Z}_2^c} &= <\bar\epsilon_1,\bar\epsilon_{3a},\bar\tau_3\bar\epsilon_2>_{\bar{\mathcal{R}}^{\mathbb{O}\oplus\mathbb{Z}_2^c}}, \quad \bar{\mathcal{M}}^{\mathbb{I}\oplus\mathbb{Z}_2^c} =<\bar\epsilon_1,\overline{\nabla\iota_6},\overline{\nabla\iota_{10}}>_{\bar{\mathcal{R}}^{\mathbb{I}\oplus\mathbb{Z}_2^c}}.
\end{aligned}$$

Similar to the invariants $\bar{\mathcal{M}}_i^L, i\in\mathbb{N}$, spans the whole space $\bar{\mathcal{M}}^L$, but the sum is no longer direct. For instance in case $L=\mathbb{T}$ the relevant subspaces for the decomposition of $\bar{\mathcal{M}}^{\mathbb{T}}$ are

$$\bar{\mathcal{S}}_i^{\mathbb{T}} :=<\bar\epsilon_1>_{\bar{\mathcal{Q}}_{i-1}^{\mathbb{O}^-}}+<\bar\epsilon_2>_{\bar{\mathcal{Q}}_{i-2}^{\mathbb{O}^-}}+<\bar\epsilon_{3a}>_{\bar{\mathcal{Q}}_{i-3}^{\mathbb{O}^-}}+<\bar\epsilon_{3b}>_{\bar{\mathcal{Q}}_{i-3}^{\mathbb{O}^-}}+<\bar\epsilon_4>_{\bar{\mathcal{Q}}_{i-4}^{\mathbb{O}^-}}+<\bar\epsilon_5>_{\bar{\mathcal{Q}}_{i-5}^{\mathbb{O}^-}} \tag{4.22}$$

with $i\geq 0$ and $\bar{\mathcal{Q}}_i^L$ defined in Subsection 4.1 ($\bar{\mathcal{Q}}_i^L:=\{0\}$ in case i is negative). Define $\bar{\mathcal{S}}_i^L$ similar for the other relevant subgroups L using their generators from Corollary 4.15 and the respective generating ring.

The interpretation of $\bar{\mathcal{S}}_i^L$ is similar to the one of $\bar{\mathcal{Q}}_i^L$: $\bar{\mathcal{S}}_i^L$ contains restrictions of polynomial mappings of degree i, but not less than i.

Proposition 4.16 *We have for $L\supset\mathbb{T}$ and $j\geq 1$*

$$\bigoplus_{i=1}^{j}\bar{\mathcal{S}}_i^L=\bar{\mathcal{M}}_{\leq j}^L \text{ as well as } \bigoplus_{i=1}^{\infty}\bar{\mathcal{S}}_i^L=\bar{\mathcal{M}}^L. \tag{4.23}$$

The sum in (4.22) (and similar for the other cases of L) is direct and the dimension of the spaces $\bar{\mathcal{S}}_i^L$ can be obtained by the coefficients of the modified Poincaré–series

$$P_{\bar{\mathcal{M}}}^L(s)=(1-s^2)\cdot P_{\mathcal{M}}^L(s).$$

Proof. We prove this proposition again only in the case $L=\mathbb{T}$, for the other cases are similar. For (4.23) it suffices to prove the first equation. $\bar{\mathcal{S}}_1^{\mathbb{T}}+\ldots+\bar{\mathcal{S}}_j^{\mathbb{T}}=\bar{\mathcal{M}}_{\leq j}^{\mathbb{T}}$ follows from $\bigoplus_{i=0}^k\bar{\mathcal{Q}}_i^{\mathbb{O}^-}=\bar{\mathcal{R}}_{\leq k}^{\mathbb{O}^-}$ (cf. Proposition 4.5). The only nontrivial statement is that the sum is direct. We claim

$$\bar{\mathcal{S}}_i^{\mathbb{T}}\cap\bar{\mathcal{S}}_j^{\mathbb{T}}=\{0\} \text{ for } i\neq j.$$

Assume there were some $\bar e\in(\bar{\mathcal{S}}_i^{\mathbb{T}}\cap\bar{\mathcal{S}}_j^{\mathbb{T}})\backslash\{0\}$. Using the index set $I:=\{1,2,3a,3b,4,5\}$ we find $\bar q_{i-\beta}\in\bar{\mathcal{Q}}_{i-\beta}^{\mathbb{O}^-}$ and $\bar p_{j-\beta}\in\bar{\mathcal{Q}}_{j-\beta}^{\mathbb{O}^-}$, $\beta\in I$ (with the obvious abuse of notation), such that

$$\bar{e} = \sum_{\beta \in I} \bar{q}_{i-\beta} \bar{\epsilon}_\beta = \sum_{\beta \in I} \bar{p}_{j-\beta} \bar{\epsilon}_\beta,$$

and at least one of the $\bar{q}'s$ and one of the $\bar{p}'s$ is nonzero. Thus $\bar{e}$ is the restriction of two homogeneous polynomial mappings e_i and e_j of degree i and j, respectively. We find $q_{i-\beta} \in \mathcal{R}^{\mathbb{O}^-}_{i-\beta}$ and $p_{j-\beta} \in \mathcal{R}^{\mathbb{O}^-}_{j-\beta}$ such that

$$e_i = \sum_{\beta \in I} q_{i-\beta} \epsilon_\beta \quad \text{and} \quad e_j = \sum_{\beta \in I} p_{j-\beta} \epsilon_\beta. \tag{4.24}$$

Now e_i homogeneous of degree i gives

$$e_i(x,y,z) = |(x,y,z)|^i \bar{e}\left(\frac{(x,y,z)}{|(x,y,z)|} \right), \tag{4.25}$$

and similar for e_j. We conclude $e_i(x,y,z) = e_j(x,y,z)|(x,y,z)|^{i-j}$ (w.l.o.g. $i > j$). Certainly $i-j$ must be an odd number, because e_i was a polynomial mapping. Therefore $k := \frac{i-j}{2} \in \mathbb{N}$ and we obtain $e_i = \rho_2^k e_j$. Together with (4.24) we get

$$\sum_{\beta \in I} \underbrace{\left(q_{i-\beta} - \rho_2^k p_{j-\beta}\right)}_{\in \mathcal{R}^{\mathbb{O}^-}} \epsilon_\beta = 0.$$

But $< \epsilon_1, \epsilon_2, \epsilon_{3a}, \epsilon_{3b}, \epsilon_4, \epsilon_5 >_{\mathcal{R}^{\mathbb{O}^-}}$ gave a minimal set of generators (cf. the Poincaré–series for $\mathcal{M}^{\mathbb{T}}$) and therefore all coefficients in the above equation must be zero: $q_{i-\beta} = \rho_2^k p_{j-\beta}$ for all $\beta \in I$. We assumed that at least one of the $\bar{p}'s$ and hence of the $p's$ is nonzero, e.g. $p_{j-\beta_0}$, giving $q_{i-\beta_0} \notin \bar{\mathcal{Q}}^{\mathbb{O}^-}_{i-\beta_0}$. This is a contradiction.

To see that the sum in (4.22) is direct we can use a similar argument as we used in (4.25). At last the statement on the Poincaré–series of $\bar{\mathcal{M}}^{\mathbb{T}}$ is now immediately clear from

$$\dim \bar{\mathcal{S}}^{\mathbb{T}}_i = \sum_{\beta \in I} \dim \bar{\mathcal{Q}}^{\mathbb{O}^-}_{i-\beta}$$

and the Poincaré–series of $\bar{\mathcal{R}}^{\mathbb{O}^-}$. $\square$

Before we continue searching an appropriate decomposition of $\bar{\mathcal{M}}^{\mathbb{T}}$, we have to introduce the canonical scalar product on $[L^2(S^2)]^3$

$$(\bar{e}, \bar{b})_{[L^2(S^2)]^3} := \sum_{i=1}^{3} (\bar{e}_{[i]}, \bar{b}_{[i]})_{L^2(S^2)}, \qquad \text{for } \bar{e}, \bar{b} \in [L^2(S^2)]^3.$$

It is easy to see that this scalar product is $\mathbf{O}(3)$ invariant: $(\gamma\bar{e},\gamma\bar{b})_{[L^2(S^2)]^3} = (\bar{e},\bar{b})_{[L^2(S^2)]^3}$ for any $\gamma \in \mathbf{O}(3)$.

Before we want to give a decomposition of $\bar{\mathcal{M}}^{\mathbb{T}}$, we start decomposing $\bar{\mathcal{M}}^{\mathbb{O}}$ and $\bar{\mathcal{M}}^{\mathbb{O}^-}$. The following two subsets of $\bar{\mathcal{M}}^{\mathbb{T}}$ will prove to be important for us. Set

$$\bar{\mathcal{N}}^{\mathbb{O}} :=< \bar{\epsilon}_4, \bar{\tau}_3\bar{\epsilon}_{3b}, \bar{\tau}_3\bar{\epsilon}_5 >_{\bar{\mathcal{R}}^{\mathbb{O}\oplus\mathbb{Z}_2^c}} \text{ and } \bar{\mathcal{N}}^{\mathbb{O}^-} :=< \bar{\epsilon}_2, \bar{\tau}_3\bar{\epsilon}_1, \bar{\tau}_3\bar{\epsilon}_{3a} >_{\bar{\mathcal{R}}^{\mathbb{O}\oplus\mathbb{Z}_2^c}} . \tag{4.26}$$

Theorem 4.17 *Let $\mathbb{T} \subset \mathbf{O}(3)$ be fixed as in Subsection 3.1.1 and let $\mathbb{O}^-$ and $\mathbb{O}\oplus\mathbb{Z}_2^c$ be supergroups of $\mathbb{T}$. Then*

$$\bar{\mathcal{M}}^{\mathbb{O}} = \bar{\mathcal{M}}^{\mathbb{O}\oplus\mathbb{Z}_2^c} \oplus \bar{\mathcal{N}}^{\mathbb{O}} \tag{4.27}$$

$$\textit{and } \bar{\mathcal{M}}^{\mathbb{O}^-} = \bar{\mathcal{M}}^{\mathbb{O}\oplus\mathbb{Z}_2^c} \oplus \bar{\mathcal{N}}^{\mathbb{O}^-} \tag{4.28}$$

holds, where both decompositions are orthogonal in $[L^2(S^2)]^3$.

Proof. From Corollary 4.15 we infer $\bar{\mathcal{M}}^{\mathbb{O}\oplus\mathbb{Z}_2^c} + \bar{\mathcal{N}}^{\mathbb{O}} = \bar{\mathcal{M}}^{\mathbb{O}}$ and for $\bar{\mathcal{M}}^{\mathbb{O}\oplus\mathbb{Z}_2^c} + \bar{\mathcal{N}}^{\mathbb{O}^-} = \bar{\mathcal{M}}^{\mathbb{O}^-}$ we use additionally $\bar{\mathcal{R}}^{\mathbb{O}^-} = \bar{\mathcal{R}}^{\mathbb{O}\oplus\mathbb{Z}_2^c} \oplus \bar{\tau}_3 \cdot \bar{\mathcal{R}}^{\mathbb{O}\oplus\mathbb{Z}_2^c}$ from Theorem 4.7.

All that remains to be shown is that both decompositions are orthogonal, because then they are direct as well. We have to show

$$(\bar{e},\bar{b})_{[L^2(S^2)]^3} = 0, \text{ for all } \bar{e} \in \bar{\mathcal{M}}^{\mathbb{O}\oplus\mathbb{Z}_2^c} \text{ and } \bar{b} \in \bar{\mathcal{N}}^{\mathbb{O}^-} \cup \bar{\mathcal{N}}^{\mathbb{O}}.$$

We have for $\gamma := -\mathbb{1} \in \mathbb{O}\oplus\mathbb{Z}_2^c$ that $\gamma\bar{b} = -\bar{b}$ for any $\bar{b} \in \bar{\mathcal{N}}^{\mathbb{O}^-} \cup \bar{\mathcal{N}}^{\mathbb{O}}$, because $\bar{b}$ is a restriction of a polynomial of even degree. On the other hand $\gamma\bar{e} = \bar{e}$ for $\bar{e} \in \bar{\mathcal{M}}^{\mathbb{O}\oplus\mathbb{Z}_2^c}$ and the theorem follows from the $\mathbf{O}(3)-$invariance of the scalar product. □

The theorem yields that the elements in $\bar{\mathcal{N}}^{\mathbb{O}^-}$ and $\bar{\mathcal{N}}^{\mathbb{O}}$ contain the 'real' $\mathbb{O}$ and $\mathbb{O}^-$ equivariant mappings. Let us now decompose $\bar{\mathcal{M}}^{\mathbb{T}}$ into spaces of more symmetry.

Theorem 4.18 *Let $\mathbb{T} \subset \mathbf{O}(3)$ be fixed as in Section 3 and $\mathbb{O} \supset \mathbb{T}$. Using $U^{\bar{\mathcal{M}}} :=< \bar{\epsilon}_{3b}, \bar{\epsilon}_5, \bar{\tau}_3\bar{\epsilon}_4 >_{\bar{\mathcal{R}}^{\mathbb{O}\oplus\mathbb{Z}_2^c}}$, we claim*

$$\bar{\mathcal{M}}^{\mathbb{T}\oplus\mathbb{Z}_2^c} = \bar{\mathcal{M}}^{\mathbb{O}\oplus\mathbb{Z}_2^c} \oplus U^{\bar{\mathcal{M}}} \tag{4.29}$$

$$\textit{and } \bar{\mathcal{M}}^{\mathbb{T}} = \bar{\mathcal{M}}^{\mathbb{O}\oplus\mathbb{Z}_2^c} \oplus \bar{\mathcal{N}}^{\mathbb{O}} \oplus \bar{\mathcal{N}}^{\mathbb{O}^-} \oplus U^{\bar{\mathcal{M}}}. \tag{4.30}$$

Both decompositions are pairwise orthogonal with respect to $(.,.)_{[L^2(S^2)]^3}$.

Proof. The above decompositions are clearly possible by Corollary 4.15.

To prove orthogonality note that Theorem 4.17 already gives $\bar{\mathcal{N}}^{\mathbb{O}} \perp \bar{\mathcal{M}}^{\mathbb{O}\oplus\mathbb{Z}_2^c}$ and $\bar{\mathcal{N}}^{\mathbb{O}^-} \perp \bar{\mathcal{M}}^{\mathbb{O}\oplus\mathbb{Z}_2^c}$. $\bar{\mathcal{N}}^{\mathbb{O}} \perp U^{\bar{\mathcal{M}}}$ and $\bar{\mathcal{N}}^{\mathbb{O}^-} \perp U^{\bar{\mathcal{M}}}$ follow by the same argument given in that proof. To see $\bar{\mathcal{N}}^{\mathbb{O}^-} \perp \bar{\mathcal{N}}^{\mathbb{O}}$ and $\bar{\mathcal{M}}^{\mathbb{O}\oplus\mathbb{Z}_2^c} \perp U^{\bar{\mathcal{M}}}$ we use an element of order 4 $\xi_4 \in \mathbb{O}\setminus\mathbb{T}$. Observe that

$$\xi_4 \bar{b} = -\bar{b} \text{ for all } \bar{b} \in \bar{\mathcal{N}}^{\mathbb{O}^-} \cup U^{\bar{\mathcal{M}}}$$

and the proof is accomplished. □

From the above theorem we conclude, that besides $U^{\bar{\mathcal{M}}}$ all components in the decomposition of $\bar{\mathcal{M}}^{\mathbb{T}}$ have actually more symmetry than only $\mathbb{T}$ or $\mathbb{T}\oplus\mathbb{Z}_2^c$. Our final goal is to separate from $U^{\bar{\mathcal{M}}} \subset \bar{\mathcal{M}}^{\mathbb{T}\oplus\mathbb{Z}_2^c}$ those mappings which have more symmetry (in the sense that they are a sum of mappings in $\bar{\mathcal{M}}^{\mathbb{O}\oplus\mathbb{Z}_2^c}$ and $\bar{\mathcal{M}}^{\mathbb{I}\oplus\mathbb{Z}_2^c}$). We proceed as in Subsection 4.1.

Similarly, let $V^{\bar{\mathcal{M}}} := \mathrm{Proj}_{U^{\bar{\mathcal{M}}}}(\bar{\mathcal{M}}^{\mathbb{I}\oplus\mathbb{Z}_2^c}) \subset U^{\bar{\mathcal{M}}}$ (with $\mathrm{Proj}_{U^{\bar{\mathcal{M}}}}$ the orthogonal projection on $U^{\bar{\mathcal{M}}}$ resulting from the decomposition (4.29)). The space $U^{\bar{\mathcal{M}}}$ decomposes orthogonally to

$$U^{\bar{\mathcal{M}}} = V^{\bar{\mathcal{M}}} \oplus W^{\bar{\mathcal{M}}}, \tag{4.31}$$

where $W^{\bar{\mathcal{M}}} := \{\bar{u} \in U^{\bar{\mathcal{M}}} \mid (\bar{u}, \bar{v})_{[L^2(S^2)]^3} = 0,\ \forall \bar{v} \in V^{\bar{\mathcal{M}}}\} = \mathrm{Proj}^{\perp}_{U^{\bar{\mathcal{M}}}}(\bar{\mathcal{M}}^{\mathbb{I}\oplus\mathbb{Z}_2^c}) \subset U^{\bar{\mathcal{M}}}$. Now

$$V^{\bar{\mathcal{M}}} \subset \mathrm{Span}\{\bar{\mathcal{M}}^{\mathbb{O}\oplus\mathbb{Z}_2^c}, \bar{\mathcal{M}}^{\mathbb{I}\oplus\mathbb{Z}_2^c}\}. \tag{4.32}$$

follows in the same fashion as (4.9) for $V^{\bar{\mathcal{R}}}$ was derived.

Again, the elements in $V^{\bar{\mathcal{M}}}$ can all be written as a sum of two polynomial mappings with the additional symmetry $\mathbb{O}\oplus\mathbb{Z}_2^c$ or $\mathbb{I}\oplus\mathbb{Z}_2^c$, respectively. Only the space $W^{\bar{\mathcal{M}}}$ seems to have pure $\mathbb{T}\oplus\mathbb{Z}_2^c$ symmetry:

Theorem 4.19 *Let $\mathbb{T} \subset \mathbf{O}(3)$ be as in Section 3 and let $\mathbb{O}$ as well as $\mathbb{I}$ be supergroups of $\mathbb{T}$ as before. Using the spaces $V^{\bar{\mathcal{M}}}$ and $W^{\bar{\mathcal{M}}}$ defined above we claim*

$$\bar{\mathcal{M}}^{\mathbb{O}\oplus\mathbb{Z}_2^c} \oplus V^{\bar{\mathcal{M}}} = \mathrm{Span}\{\bar{\mathcal{M}}^{\mathbb{O}\oplus\mathbb{Z}_2^c}, \bar{\mathcal{M}}^{\mathbb{I}\oplus\mathbb{Z}_2^c}\}. \tag{4.33}$$

Consequently,

$$W^{\bar{\mathcal{M}}} \perp \mathrm{Span}\{\bar{\mathcal{M}}^{\mathbb{O}\oplus\mathbb{Z}_2^c}, \bar{\mathcal{M}}^{\mathbb{I}\oplus\mathbb{Z}_2^c}\} \tag{4.34}$$

$$\textit{and } \bar{\mathcal{M}}^{\mathbb{T}\oplus\mathbb{Z}_2^c} = \mathrm{Span}\{\bar{\mathcal{M}}^{\mathbb{O}\oplus\mathbb{Z}_2^c}, \bar{\mathcal{M}}^{\mathbb{I}\oplus\mathbb{Z}_2^c}\} \oplus W^{\bar{\mathcal{M}}} \tag{4.35}$$

holds. Furthermore, $W^{\bar{\mathcal{M}}}$ is independent of the particular choice of $\mathbb{I} \supset \mathbb{T}$ (cf. Subsubsection 3.1.6).

Proof. We omit this proof, because it can be done along the lines of the proof of Theorem 4.8, where we have shown the analogous statement for $W^{\bar{\mathcal{R}}}$. □

Again elements in $W^{\bar{\mathcal{M}}}$ will be of major interest to us, since they contain all elements with precise $\mathbb{T}\oplus\mathbb{Z}_2^c$ symmetry. The following definition provides subspaces, which eventually give the decomposition of $W^{\bar{\mathcal{M}}}$. For the rest of this section we are only interested in polynomial mappings which are at least $\mathbb{T}\oplus\mathbb{Z}_2^c-$equivariant. Observe that the elements of $\bar{\mathcal{M}}^{\mathbb{T}\oplus\mathbb{Z}_2^c}$ are all restrictions of polynomials of *odd* degree (cf. Corollary 4.15), so we do not have to worry about any even degree polynomial mappings.

Definition 4.20 *Let $W_{2j+1}^{\bar{\mathcal{M}}}$, $j \geq 0$ be recursively defined as the maximal subspace of $\bar{\mathcal{M}}_{2j+1}^{\mathbb{T}\oplus\mathbb{Z}_2^c} \cap U^{\bar{\mathcal{M}}} = \bar{\mathcal{M}}_{\leq 2j+1}^{\mathbb{T}\oplus\mathbb{Z}_2^c} \cap U^{\bar{\mathcal{M}}} \subset U^{\bar{\mathcal{M}}}$ which satisfies the condition*

$$W_{2j+1}^{\bar{\mathcal{M}}} \perp \operatorname{Span}\{\bar{\mathcal{M}}_{\leq 2j+1}^{\mathbb{I}\oplus\mathbb{Z}_2^c}, \bigoplus_{i=0}^{j-1} W_{2i+1}^{\bar{\mathcal{M}}}\}. \tag{4.36}$$

Some of these subspaces will only contain 0, and therefore these subspaces won't contribute much to our decomposition. Theorem 4.22 will tell us exactly which of them. We have:

Theorem 4.21 *$(W_{2j+1}^{\bar{\mathcal{M}}})_{j\geq 0}$ is a sequence of pairwise orthogonal subspaces in $[L^2(S^2)]^3$ which satisfy*

$$W_{2j+1}^{\bar{\mathcal{M}}} \perp \operatorname{Span}\{\bar{\mathcal{M}}^{\mathbb{O}\oplus\mathbb{Z}_2^c}, \bar{\mathcal{M}}^{\mathbb{I}\oplus\mathbb{Z}_2^c}\}. \tag{4.37}$$

In particular, they form an orthogonal decomposition of $W^{\bar{\mathcal{M}}}$:

$$W^{\bar{\mathcal{M}}} = \bigoplus_{j=0}^{\infty} W_{2j+1}^{\bar{\mathcal{M}}}. \tag{4.38}$$

Proof. Once more we make use of the proof of the analogous theorem in Subsection 4.1. Replace just the projection $Q_{\bar{\mathcal{R}}}^L$ onto $\bar{\mathcal{R}}^L$ by

$$Q_{\bar{\mathcal{M}}}^L(\bar{e}) := \operatorname{Proj}_{\bar{\mathcal{M}}^L}(\bar{e}) = \frac{1}{|L|}\sum_{\gamma\in L}\gamma\bar{e} \in \bar{\mathcal{M}}^L, \text{ for } \bar{e} \in \bar{\mathcal{M}},$$

using the action (4.21) of L on $\bar{\mathcal{M}}$. Everything else works out as before, now with the new scalar product on $[L^2(S^2)]^3$. □

The next theorem in this section will tell us how large $W_{2j+1}^{\bar{\mathcal{M}}}$ actually is.

Theorem 4.22 *We obtain for any $j \geq 0$*

$$\bar{\mathcal{M}}^{\mathbb{T}\oplus\mathbb{Z}_2^c}_{\leq 2j+1} = \mathrm{Span}\left\{\bar{\mathcal{M}}^{\mathbb{O}\oplus\mathbb{Z}_2^c}_{\leq 2j+1}, \bar{\mathcal{M}}^{\mathbb{I}\oplus\mathbb{Z}_2^c}_{\leq 2j+1}\right\} \oplus \bigoplus_{i=0}^{j} W^{\bar{\mathcal{M}}}_{2i+1}. \tag{4.39}$$

Furthermore, the dimension of $W^{\bar{\mathcal{M}}}_{2j+1}$ is given by the coefficient of s^{2j+1} in the Poincaré–series

$$\begin{aligned} P^W_{\bar{\mathcal{M}}}(s) &:= \frac{s^3\left(1+s^6+s^{12}\right)}{\left(1-s^4\right)\left(1-s^{10}\right)} \\ &= s^3+s^7+s^9+s^{11}+2\,s^{13}+2\,s^{15}+2\,s^{17}+3\,s^{19}+O\left(s^{21}\right). \end{aligned} \tag{4.40}$$

Proof. Equation (4.39) follows again from (4.35) and (4.38) by projecting both sides to $\bar{\mathcal{M}}^{\mathbb{T}\oplus\mathbb{Z}_2^c}_{\leq 2j+1}$.

Proposition 4.16 gives the decomposition $\bar{\mathcal{M}}^{\mathbb{T}\oplus\mathbb{Z}_2^c}_{\leq 2j+1} = \bigoplus_{i=0}^{j} \bar{\mathcal{S}}^{\mathbb{T}\oplus\mathbb{Z}_2^c}_{2i+1}$ (note that $\bar{\mathcal{S}}^{\mathbb{T}\oplus\mathbb{Z}_2^c}_{2i} = \{0\}$) and similar ones for $\bar{\mathcal{M}}^{\mathbb{O}\oplus\mathbb{Z}_2^c}_{\leq 2j+1}$ and $\bar{\mathcal{M}}^{\mathbb{I}\oplus\mathbb{Z}_2^c}_{\leq 2j+1}$. With an argument very similar to the one given in the proof of Theorem 4.11, we find for $j \geq 0$:

$$\dim W^{\bar{\mathcal{M}}}_{2j+1} = \dim \bar{\mathcal{S}}^{\mathbb{T}\oplus\mathbb{Z}_2^c}_{2j+1} - (\dim \bar{\mathcal{S}}^{\mathbb{O}\oplus\mathbb{Z}_2^c}_{2j+1} + \dim \bar{\mathcal{S}}^{\mathbb{I}\oplus\mathbb{Z}_2^c}_{2j+1} - \dim \bar{\mathcal{S}}^{\mathbf{O}(3)}_{2j+1}).$$

The Poincaré–series of $\mathbf{O}(3)$ for the module is $P^{\mathbf{O}(3)}_{\mathcal{M}}(s) = \frac{s}{1-s^2}$ and hence $P^{\mathbf{O}(3)}_{\bar{\mathcal{M}}}(s) = s$ giving $\dim \bar{\mathcal{S}}^{\mathbf{O}(3)}_{2j+1}$. Using Proposition 4.16, $\dim W^{\bar{\mathcal{M}}}_{2j+1}$ is therefore given by the $(2j+1)$–th coefficient of the Poincaré–series

$$\begin{aligned} P^W_{\bar{\mathcal{M}}}(s) &= P^{\mathbb{T}\oplus\mathbb{Z}_2^c}_{\bar{\mathcal{M}}}(s) - (P^{\mathbb{O}\oplus\mathbb{Z}_2^c}_{\bar{\mathcal{M}}}(s) + P^{\mathbb{I}\oplus\mathbb{Z}_2^c}_{\bar{\mathcal{M}}}(s) - P^{\mathbf{O}(3)}_{\bar{\mathcal{M}}}(s)) \\ &= \frac{s+2\,s^5+2\,s^3+s^7}{\left(1-s^4\right)\left(1-s^6\right)} - \frac{s+s^3+s^5}{\left(1-s^4\right)\left(1-s^6\right)} - \frac{s+s^5+s^9}{\left(1-s^6\right)\left(1-s^{10}\right)} + s \\ &= \frac{s^3\left(1+s^6+s^{12}\right)}{\left(1-s^4\right)\left(1-s^{10}\right)}. \end{aligned}$$

□

Note that again $W^{\bar{\mathcal{M}}}$ is not an algebra. We find e.g. $W^{\bar{\mathcal{M}}}_3 = \mathrm{Span}\{\bar{w}^{\bar{\mathcal{M}}}_3\}$ and $W^{\bar{\mathcal{M}}}_7 = \mathrm{Span}\{\bar{w}^{\bar{\mathcal{M}}}_7\}$ with

$$\bar{w}^{\bar{\mathcal{M}}}_3 := \bar{\epsilon}_{3b} \text{ and } \bar{w}^{\bar{\mathcal{M}}}_7 := -\frac{15}{11}\bar{\rho}_2^2\bar{\epsilon}_{3b} + 3\bar{\rho}_4\bar{\epsilon}_{3b} + 12\bar{\tau}_3\bar{\epsilon}_4.$$

Since $\bar{w}_3^{\bar{\mathcal{M}}}$ and $\bar{w}_7^{\bar{\mathcal{M}}}$ are obviously in $U^{\bar{\mathcal{M}}}$ we only have to check $\bar{w}_3^{\bar{\mathcal{M}}} \perp \bar{\mathcal{M}}_{\leq 3}^{\mathbb{I}\oplus\mathbb{Z}_2^c} = \mathrm{Span}\{\bar{\epsilon}_1\}$ and $\bar{w}_7^{\bar{\mathcal{M}}} \perp \mathrm{Span}\{\bar{\mathcal{M}}_{\leq 7}^{\mathbb{I}\oplus\mathbb{Z}_2^c}, W_3^{\bar{\mathcal{M}}}\} = \mathrm{Span}\{\bar{\epsilon}_1, \overline{\nabla\iota_6}, \bar{\iota}_6\bar{\epsilon}_1, \bar{w}_3^{\bar{\mathcal{M}}}\}$.

Some more structure of the space $W^{\bar{\mathcal{M}}}$ can be seen in the last theorem of this section.

Theorem 4.23 *$W^{\bar{\mathcal{R}}}$ is embedded in $W^{\bar{\mathcal{M}}}$ in the following sense:*

$$W^{\bar{\mathcal{R}}}\bar{\epsilon}_1 \subset W^{\bar{\mathcal{M}}} \text{ and } \nabla W^{\bar{\mathcal{R}}} := \{\overline{\nabla p} \mid p \in \mathcal{R},\ p \text{ is homogeneous and } p_{|S^2} = \bar{p} \in W^{\bar{\mathcal{R}}}\} \subset W^{\bar{\mathcal{M}}}.$$

Proof. We have $\bar{e} \in W^{\bar{\mathcal{R}}}\bar{\epsilon}_1 \subset \bar{\mathcal{M}}^{\mathbb{T}\oplus\mathbb{Z}_2^c}$, since $W^{\bar{\mathcal{R}}} \subset \bar{\mathcal{R}}^{\mathbb{T}\oplus\mathbb{Z}_2^c}$. All we have to show by Theorem 4.19 is

$$\bar{e} \perp \mathrm{Span}\{\bar{\mathcal{M}}^{\mathbb{O}\oplus\mathbb{Z}_2^c}, \bar{\mathcal{M}}^{\mathbb{I}\oplus\mathbb{Z}_2^c}\},$$

or, equivalently, we have to show $Q_{\bar{\mathcal{M}}}^L(\bar{e}) = 0$ for $L = \mathbb{O}\oplus\mathbb{Z}_2^c$ and $\mathbb{I}\oplus\mathbb{Z}_2^c$. But this follows from $W^{\bar{\mathcal{R}}} \perp \mathrm{Span}\{\bar{\mathcal{R}}^{\mathbb{O}\oplus\mathbb{Z}_2^c}, \bar{\mathcal{R}}^{\mathbb{I}\oplus\mathbb{Z}_2^c}\}$:

$$Q_{\bar{\mathcal{M}}}^L(\bar{e}) = Q_{\bar{\mathcal{M}}}^L(\bar{w}^{\bar{\mathcal{R}}}\bar{\epsilon}_1) = Q_{\bar{\mathcal{R}}}^L(\bar{w}^{\bar{\mathcal{R}}})\bar{\epsilon}_1 = 0.$$

The second assertion is an immediate consequence of

$$Q_{\bar{\mathcal{M}}}^L(\nabla p) = \nabla Q_{\bar{\mathcal{R}}}^L(p) \text{ for } p \in \mathcal{R}.$$

□

5 Parametrization of the Fixed-Point Subspaces

In the sequel we derive parametrizations for elements of $\Upsilon \in \mathcal{H}_{(L,G/H)}$ in the case $G = \mathbf{O}(3)$, $H = \mathbf{O}(2)\oplus\mathbb{Z}_2^c$ or $\mathbf{O}(2)^-$ and L a supergroup of $\mathbb{T}$. These parametrizations will be necessary to evaluate the flow formula (2.22) in the next section.

We assume that the kernel $\ker A(\lambda_0) \subset L^2(S^2)$ is irreducible for the given (standard) $\mathbf{O}(3)-$action. This assumption will guarantee easy parametrizations of the relevant connections. To see that, we introduce the space $\mathcal{SH}_l \subset L^2(S^2)$ of spherical harmonics in three variables and of degree $l \in \mathbb{N}_0$. It is well known that any irreducible representation of $\mathbf{O}(3)$ is isomorphic to the (minus or plus) representation of $\mathbf{O}(3)$ on $\mathcal{SH}_l$, for some l (see for instance [5], Chapter XIII Theorem 7.5).

Our special situation, however, is even better. Since $\ker A(\lambda_0)$ is already a subspace of $L^2(S^2)$, we claim that $\ker A(\lambda_0)$ is actually equal to some $\mathcal{SH}_l$ (equipped with the standard action). The restriction of the standard representation of $\mathbf{O}(3)$ on $L^2(S^2)$ to $\mathcal{SH}_l$ is usually called the natural representation of $\mathbf{O}(3)$ on $\mathcal{SH}_l$. This is the minus representation for l odd and the plus representation for l even (see [5] Chapter XIII §9 (e)).

Lemma 5.1 *Let* $\{0\} \neq V \subset L^2(S^2)$ *be an irreducible representation for the standard action of* $\mathbf{O}(3)$*. Then*

$$V = \mathcal{SH}_{l_0}, \quad \textit{for some} \quad l_0 \in \mathbb{N}_0.$$

Furthermore the $\mathbf{O}(3)-$*module* V *is equal to the* $\mathbf{O}(3)-$*module* $\mathcal{SH}_{l_0}$*, where* $\mathbf{O}(3)$ *is acting naturally.*

Proof. Consider the orthogonal projections onto $\mathcal{SH}_l$, i.e. $P_l : L^2(S^2) \to \mathcal{SH}_l \subset L^2(S^2)$. They are obviously $\mathbf{O}(3)-$equivariant. Due to the irreducibility of V and $\mathcal{SH}_l$ it follows that the restriction $P_{l|V} : V \to \mathcal{SH}_l$ is either trivial or an $\mathbf{O}(3)-$equivariant isomorphism. Since V was not trivial and $L^2(S^2) = \bigoplus_{l=0}^{\infty} \mathcal{SH}_l$ (see [18] pp. 436-457) we derive that there is at least one $l_0 \in \mathbb{N}_0$, such that $V \cong \mathcal{SH}_{l_0}$ via P_{l_0}. On the other hand $\dim(\mathcal{SH}_l) = 2l+1$ gives that l_0 is the only $l \in \mathbb{N}_0$ with $P_{l|V}$ is nontrivial. Hence,

$$\mathcal{SH}_{l_0} \cong V = P_{l_0}(V) \subset \mathcal{SH}_{l_0}$$

giving $V = \mathcal{SH}_{l_0}$. Therefore, $P_{l_0|V}$ is just the identity and V as an $\mathbf{O}(3)-$module is equal to $\mathcal{SH}_{l_0}$ as $\mathbf{O}(3)-$module equipped with the standard action, which is the natural representation of $\mathbf{O}(3)$ on $\mathcal{SH}_{l_0}$. □

We now consider axisymmetric elements in $\mathcal{SH}_l$. If $\mathbf{SO}(2) \subset \mathbf{O}(3)$ is fixed, there is (up to multiples) only one axisymmetric spherically harmonic polynomial of degree l. Choosing $\mathbf{SO}(2)$ rotating about the $x-$axis, this polynomial is given by (cf. for instance [9], Theorem 2.4.6)

$$\begin{aligned} u_l^* = u_l^*(x,y,z) \;&:=\; \sum_{\nu=0}^{[\frac{l}{2}]} (-1)^\nu q_\nu x^{l-2\nu}(z^2+y^2)^\nu \\ q_0 = 1, \qquad & 4\nu^2 q_\nu = (l-2\nu+2)(l-2\nu+1)q_{\nu-1}, \quad \nu \geq 1. \end{aligned} \tag{5.1}$$

Obviously,

$$\Sigma_{u_l^*} = \begin{cases} \mathbf{O}(2) \oplus \mathbb{Z}_2^c & \text{for } l \text{ even} \\ \mathbf{O}(2)^- & \text{for } l \text{ odd} \end{cases} . \tag{5.2}$$

The group orbit $\mathcal{O}(u_l^*) \subset \mathcal{SH}_l$ is isomorphic to $\mathbf{O}(3)/\Sigma_{u_l^*} = \mathbf{O}(3)/H$. In the two relevant cases for H we have

$$\mathbf{O}(3)/(\mathbf{O}(2) \oplus \mathbb{Z}_2^c) \;\cong\; \mathbb{P}^2 \tag{5.3}$$

$$\mathbf{O}(3)/\mathbf{O}(2)^- \;\cong\; S^2. \tag{5.4}$$

In order to parametrize connections Υ of $(L, G/H)$ we search for an injective curve $\gamma : (0, \varphi^*) \to \mathbf{O}(3)$ such that

$$\omega_l(\varphi) := \gamma(\varphi) u_l^* \in \mathcal{SH}_l$$

parametrizes a one–dimensional subset of $Fix_{\mathcal{O}(u_l^*)}(L') \cong Fix_{G/H}(L')$, $L' \subset L$, which connects two elements of $\mathcal{E}_{(L,G/H)}$ (cf. (1.12)). The following subsections will provide such parametrizations for the various cases of $L \supset \mathbb{T}$.

Although we do not calculate the fixed-point spaces in detail, we remark that we make use of the subnormalizer $N_G(L, H) := \{\gamma \in G \mid L \subset \gamma H \gamma^{-1}\}$ (cf. [8]). It was shown in [12], Proposition 1.7, that

$$Fix_{G/H}(L') = N_G(L', H)/H \subset G/H$$

holds (see also [11] for a different way to calculate $Fix_{G/H}(L')$). We are, however, interested in the particular fixed-point space $Fix_{\mathcal{O}(u_l^*)}(L') \subset \ker A(\lambda_0) = \mathcal{SH}_l$, where $\mathcal{O}(u_l^*) \cong \mathbf{O}(3)/H$ and $\Sigma_{u_l^*} = H$. We find

$$Fix_{\mathcal{O}(u_l^*)}(L') = N_G(L', H) u_l^* \subset \mathcal{O}(u_l^*) \subset \mathcal{SH}_l.$$

5.1 The Fixed-Point Subspaces for $L = \mathbb{T}, \mathbb{T}\oplus\mathbb{Z}_2^c$ and for $H = \mathbf{O}(2)\oplus\mathbb{Z}_2^c$, $\mathbf{O}(2)^-$

We start discussing the case $L = \mathbb{T}$ and $H = \mathbf{O}(2)\oplus\mathbb{Z}_2^c$. Since

$$\mathbf{O}(3)/(\mathbf{O}(2)\oplus\mathbb{Z}_2^c) \cong \mathbb{P}^2 \cong \mathbf{SO}(3)/\mathbf{O}(2), \tag{5.5}$$

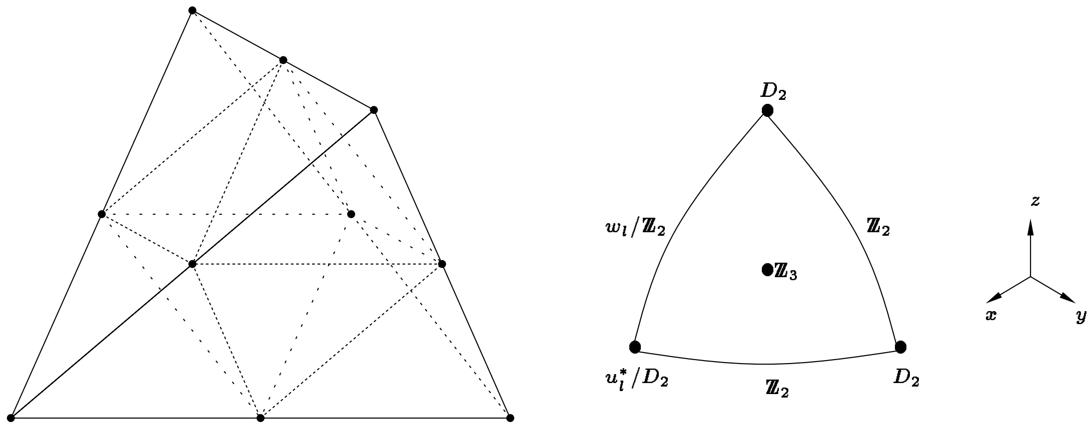
this is clearly almost the same example as given at the end of Section 1. The subgroups of $\mathbb{T}$ with nontrivial fixed-point subspace are $L' = \mathbb{Z}_2, D_2$ and $\mathbb{Z}_3$ with

$$Fix_{\mathbf{O}(3)/(\mathbf{O}(2)\oplus\mathbb{Z}_2^c)}(\mathbb{Z}_2) \cong S^1 \cup 1pt, \qquad Fix_{\mathbf{O}(3)/(\mathbf{O}(2)\oplus\mathbb{Z}_2^c)}(D_2) \cong 3pt$$
$$Fix_{\mathbf{O}(3)/(\mathbf{O}(2)\oplus\mathbb{Z}_2^c)}(\mathbb{Z}_3) \cong 1pt.$$

The set $Fix_{(\mathbb{T},\mathbf{O}(3)/(\mathbf{O}(2)\oplus\mathbb{Z}_2^c))}$, defined as the union of all the nontrivial fixed-point spaces (cf. (1.14)) is depicted in Figure 2.

Using

Figure 2: $Fix_{(\mathbb{T},\mathbf{O}(3)/(\mathbf{O}(2)\oplus\mathbb{Z}_2^c))}$



$$\gamma_\omega(\varphi) = \begin{pmatrix} \cos(\varphi) & 0 & -\sin(\varphi) \\ 0 & 1 & 0 \\ \sin(\varphi) & 0 & \cos(\varphi) \end{pmatrix},$$

we find the parametrization for the connection of $(\mathbb{T}, \mathbf{O}(3)/(\mathbf{O}(2)\oplus\mathbb{Z}_2^c))$ between u_l^* and the spherically harmonic function of (even) degree l which is axisymmetric with respect to the z-axis. Both equilibria which are connected by this branch lie (identify $Fix_{\mathcal{O}(u_l^*)}(D_2)$, l even, with $Fix_{\mathbf{O}(3)/(\mathbf{O}(2)\oplus\mathbb{Z}_2^c)}(D_2)$) in $Fix_{\mathbf{O}(3)/(\mathbf{O}(2)\oplus\mathbb{Z}_2^c)}(D_2)$. For $l = 2$ this gives a branch between $u_2^* = 2\,x^2 - (y^2 + z^2)$ and $2\,z^2 - (y^2 + x^2)$. For $\varphi \in (0, \frac{\pi}{2})$ let

$$\omega_2(\varphi) := \gamma_\omega(\varphi) u_2^* = (2 - 3\sin^2(\varphi))x^2 + (2 - 3\cos^2(\varphi))z^2 + 6\,\cos(\varphi)\sin(\varphi)xz - y^2. \quad (5.6)$$

The other connections between equilibria in $Fix_{\mathbf{O}(3)/(\mathbf{O}(2)\oplus\mathbb{Z}_2^c)}(D_2)$ cannot give any new information concerning the flow, because all connections lie on the same $\mathbb{T}-$orbit. We will not make use of other even l parametrizations, because the computational effort we have to make in Section 6 rises quickly. Nevertheless, for small l it would still be possible to obtain similar results for higher dimensional representations of the kernel.

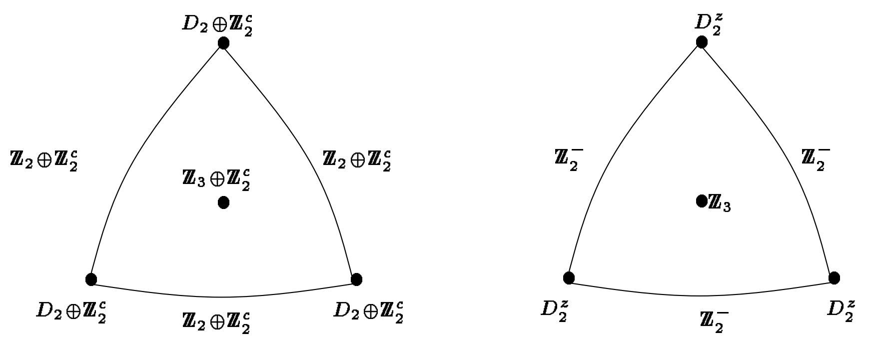
Considering $L = \mathbb{T}\oplus\mathbb{Z}_2^c$ instead of $L = \mathbb{T}$, nothing really new happens. Some new subgroups of $\mathbb{T}\oplus\mathbb{Z}_2^c$ are of the form L' or $L'\oplus\mathbb{Z}_2^c$, where L' is a subgroup of $\mathbb{T}$. There are, however, also two class III subgroups in $\mathbb{T}\oplus\mathbb{Z}_2^c$ (cf. [5], XIII Section 9 for the class III subgroups of $\mathbf{O}(3)$). The first is $\mathbb{Z}_2^- = \{\mathbb{1}, -\xi_2\}$, where ξ_2 is the generator of some $\mathbb{Z}_2 \subset \mathbb{T}$. The second is $D_2^z = \mathbb{Z}_2 \cup \{-\gamma, \gamma \in D_2\backslash\mathbb{Z}_2\}$ with again $\mathbb{Z}_2 \subset D_2 \subset \mathbb{T}$.

Since all elements in $\mathbf{O}(3)/(\mathbf{O}(2)\oplus\mathbb{Z}_2^c)$ are invariant under $\gamma = -\mathbb{1}$ one gets for $L' \subset \mathbb{T}$

$$Fix_{\mathbf{O}(3)/(\mathbf{O}(2)\oplus\mathbb{Z}_2^c)}(L') = Fix_{\mathbf{O}(3)/(\mathbf{O}(2)\oplus\mathbb{Z}_2^c)}(L'\oplus\mathbb{Z}_2^c)$$

and $Fix_{(\mathbb{T}\oplus\mathbb{Z}_2^c,\mathbf{O}(3)/(\mathbf{O}(2)\oplus\mathbb{Z}_2^c))}$ in Figure 3 follows easily.

Figure 3: $Fix_{(\mathbb{T}\oplus\mathbb{Z}_2^c,\mathbf{O}(3)/(\mathbf{O}(2)\oplus\mathbb{Z}_2^c))}$ and $Fix_{(\mathbb{T}\oplus\mathbb{Z}_2^c,\mathbf{O}(3)/\mathbf{O}(2)^-)}$



The parametrization from above is sufficient for this case as well. Considering $H = \mathbf{O}(2)^-$ we have $\mathbf{O}(3)/\mathbf{O}(2)^- \cong S^2$ and the nontrivial fixed-point subspaces are for $L' = \mathbb{Z}_2^-, D_2^z$ and $\mathbb{Z}_3$: $Fix_{\mathbf{O}(3)/\mathbf{O}(2)^-}(\mathbb{Z}_2^-) \cong S^1$, $Fix_{\mathbf{O}(3)/\mathbf{O}(2)^-}(D_2^z) \cong 2pt$ and $Fix_{\mathbf{O}(3)/\mathbf{O}(2)^-}(\mathbb{Z}_3) \cong 2pt$ (cf. Figure 3). There is, of course, also a nontrivial fixed-point subspace for $\mathbb{Z}_2 \subset D_2^z$. However, it is the same as the one for D_2^z, and therefore not worth mentioning.

We do need a new parametrization for the connections of $(\mathbb{T}\oplus\mathbb{Z}_2^c,\, \mathbf{O}(3)/\mathbf{O}(2)^-)$, since the isotropy subgroup $\Sigma_{u_l^*} = \mathbf{O}(2)^-$ is only possible for odd l. In the case $l = 3$ we get the branch between $u_3^* = 2\,x^3 - 3\,x(y^2+z^2)$ and $2\,z^3 - 3\,z(y^2+x^2)$ as

$$\begin{aligned}\omega_3(\varphi) := \gamma_\omega(\varphi)u_3^* \;=\;& \left(-3+5\,\cos^2(\varphi)\right)\cos(\varphi)\,x^3 + \left(2-5\,\cos^2(\varphi)\right)\sin(\varphi)\,z^3 \\ & +3\,\left(-1+5\,\cos^2(\varphi)\right)\sin(\varphi)\,x^2z + 3\,\left(4-5\,\cos^2(\varphi)\right)\cos(\varphi)\,xz^2 \\ & -3\,y^2(\cos(\varphi)x+\sin(\varphi)z), \quad \varphi \in (0, \frac{\pi}{2}).\end{aligned} \tag{5.7}$$

Using $\varphi^*_{\omega,\mathbb{T}} := \frac{\pi}{2}$ we denote the above constructed connections by

$$\Upsilon_l^{\omega,\mathbb{T}} := \{\omega_l(\varphi), \varphi \in (0, \varphi^*_{\omega,\mathbb{T}})\}. \tag{5.8}$$

In the last case $L = \mathbb{T}$ and $H = \mathbf{O}(2)^-$ we find only $Fix_{\mathbf{O}(3)/\mathbf{O}(2)^-}(\mathbb{Z}_2) \cong 2pt$ and furthermore $Fix_{\mathbf{O}(3)/\mathbf{O}(2)^-}(\mathbb{Z}_3) \cong 2pt$ remains left. That means there are no connections of $(\mathbb{T}, \mathbf{O}(3)/\mathbf{O}(2)^-)$.

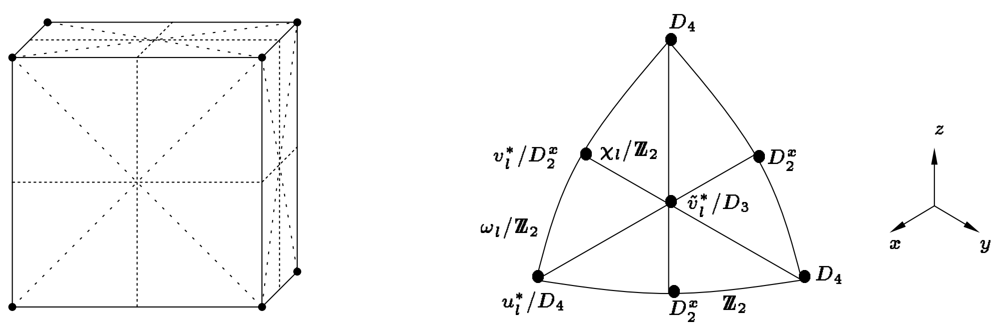
5.2 The Fixed-Point Subspaces for $L = \mathbb{O}, \mathbb{O}^-, \mathbb{O}\oplus\mathbb{Z}_2^c$ and for $H = \mathbf{O}(2)\oplus\mathbb{Z}_2^c$, $\mathbf{O}(2)^-$

We begin discussing $L = \mathbb{O}$ and $H = \mathbf{O}(2)\oplus\mathbb{Z}_2^c$. Once more $\mathbf{O}(3)/(\mathbf{O}(2)\oplus\mathbb{Z}_2^c) \cong \mathbb{P}^2 \cong \mathbf{SO}(3)/\mathbf{O}(2)$ reduces our problem to something known (cf. [12], Table 1).

The subgroups of $\mathbb{O}$ with nontrivial fixed-point subspace are $L' = \mathbb{Z}_2, D_2^x, D_4$ and D_3 (we denote by D_2^x the D_2 subgroup of $\mathbb{O}$ which is not normal in $\mathbb{O}$; this is equivalent to $D_2^x \not\subset D_4 \subset \mathbb{O}$). It follows:

$$Fix_{\mathbf{O}(3)/(\mathbf{O}(2)\oplus\mathbb{Z}_2^c)}(\mathbb{Z}_2) \cong S^1 \cup 1pt, \qquad Fix_{\mathbf{O}(3)/(\mathbf{O}(2)\oplus\mathbb{Z}_2^c)}(D_2^x) \cong 3pt$$
$$Fix_{\mathbf{O}(3)/(\mathbf{O}(2)\oplus\mathbb{Z}_2^c)}(D_3) \cong 1pt, \qquad Fix_{\mathbf{O}(3)/(\mathbf{O}(2)\oplus\mathbb{Z}_2^c)}(D_4) \cong 1pt.$$

Figure 4: $Fix_{(\mathbb{O},\mathbf{O}(3)/(\mathbf{O}(2)\oplus\mathbb{Z}_2^c))}$



As a first connection of $(\mathbb{O}, \mathbf{O}(3)/(\mathbf{O}(2)\oplus\mathbb{Z}_2^c))$ we find a subset of $\Upsilon_l^{\omega,\mathbb{T}}$: With $\varphi^*_{\omega,\mathbb{O}} := \frac{\pi}{4}$ we have

$$\Upsilon_l^{\omega,\mathbb{O}} := \{\omega_l(\varphi), \varphi \in (0, \varphi^*_{\omega,\mathbb{O}})\}, \tag{5.9}$$

which connects the equilibria in $Fix_{\mathbf{O}(3)/(\mathbf{O}(2)\oplus\mathbb{Z}_2^c)}(D_4)$ and $Fix_{\mathbf{O}(3)/(\mathbf{O}(2)\oplus\mathbb{Z}_2^c)}(D_2^x)$, i.e. for $l = 2$ the connection from $2\,x^2 - (y^2 + z^2)$ to $v_2^* := \frac{1}{2}(x^2 + z^2) + 3\,xz - y^2$.

Although for $H = \mathbf{O}(2)\oplus\mathbb{Z}_2^c$ only the representation for even l is present, we similarly intend to treat the odd l case, which we will need for connections with $H = \mathbf{O}(2)^-$ (cf. Figure 7). For $l = 3$ this will give a connection between $2\,x^3 - 3\,x(y^2 + z^2)$ and $v_3^* := -\frac{\sqrt{2}}{4}\,(x^3 + z^3 - 9\,xz(x + z) + 6\,y^2(x + z))$.

There are two more essentially different connections of $(\mathbb{O}, \mathbf{O}(3)/(\mathbf{O}(2)\oplus\mathbb{Z}_2^c))$. The second branch connects an equilibrium in $Fix_{\mathbf{O}(3)/(\mathbf{O}(2)\oplus\mathbb{Z}_2^c)}(D_2^x)$ to an equilibrium which lies in $Fix_{\mathbf{O}(3)/(\mathbf{O}(2)\oplus\mathbb{Z}_2^c)}(D_3)$. For $l = 2$ this equilibrium is $\tilde{v}_2^* := 2(xy + xz + yz)$ and for $l = 3$

the related equilibrium will be $\tilde{v}_3^* := -\frac{2\sqrt{3}}{9}(2\,(x^3+y^3+z^3) - 3\,x^2(y+z) - 3\,y^2(x+z) - 3\,z^2(x+y) - 15\,xyz)$. The corresponding branches are parametrized by

$$\gamma_\chi(\varphi) = \begin{pmatrix} \frac{\sqrt{2}}{2} & 0 & -\frac{\sqrt{2}}{2} \\ 0 & 1 & 0 \\ \frac{\sqrt{2}}{2} & 0 & \frac{\sqrt{2}}{2} \end{pmatrix} \begin{pmatrix} \cos(\varphi) & -\sin(\varphi) & 0 \\ \sin(\varphi) & \cos(\varphi) & 0 \\ 0 & 0 & 1 \end{pmatrix} \begin{pmatrix} \frac{\sqrt{2}}{2} & 0 & \frac{\sqrt{2}}{2} \\ 0 & 1 & 0 \\ -\frac{\sqrt{2}}{2} & 0 & \frac{\sqrt{2}}{2} \end{pmatrix}.$$

We set

$$\Upsilon_l^{\chi,\mathbb{O},a} := \{\chi_l(\varphi), \varphi \in (0, \varphi^*_{\chi,\mathbb{O}})\}, \tag{5.10}$$

with $\varphi^*_{\chi,\mathbb{O}} := \arccos(\frac{\sqrt{6}}{3})$ and

$$\begin{aligned} \chi_2(\varphi) := \gamma_\chi(\varphi) v_2^* \;&=\; \left(2 - 3\,\cos^2(\varphi)\right)(y^2 - \frac{1}{2}(x^2+z^2)) \\ &\quad +3\,\cos(\varphi)\sqrt{2}\sin(\varphi) y(x+z) + 3\,\cos^2(\varphi) xz. \end{aligned} \tag{5.11}$$

$$\begin{aligned} \chi_3(\varphi) \;:=\;& \gamma_\chi(\varphi) v_3^* = \frac{\sqrt{2}}{4}\left(-6 + 5\,\cos^2(\varphi)\right)\cos(\varphi)(x^3+z^3) + \left(2 - 5\,\cos^2(\varphi)\right)\sin(\varphi) y^3 \\ +\;& \frac{3}{2}\left(-2 + 5\,\cos^2(\varphi)\right)\sin(\varphi) y(x^2+z^2) + \frac{3\sqrt{2}}{4}\left(-2 + 5\,\cos^2(\varphi)\right)\cos(\varphi) zx(x+z) \\ +\;& \frac{3\sqrt{2}}{2}((4 - 5\,\cos^2(\varphi))\cos(\varphi) y^2(x+z) + 15\,\cos^2(\varphi)\sin(\varphi) xyz. \end{aligned} \tag{5.12}$$

The last connection between the equilibria in $Fix_{\mathbf{O}(3)/(\mathbf{O}(2)\oplus\mathbb{Z}_2^c)}(D_3)$ and the equilibria in $Fix_{\mathbf{O}(3)/(\mathbf{O}(2)\oplus\mathbb{Z}_2^c)}(D_4)$ is also obtained by χ_l.
We take

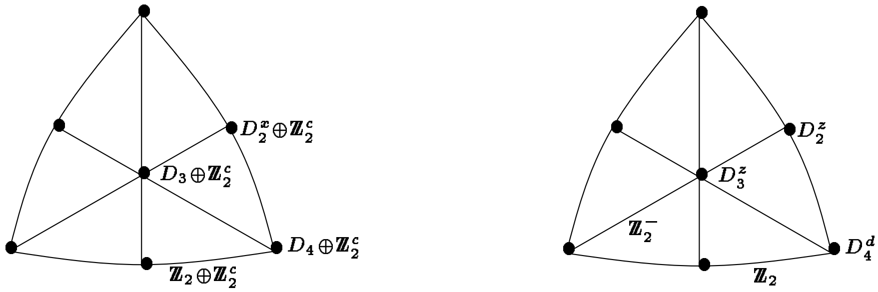
$$\Upsilon_l^{\chi,\mathbb{O},b} := \{\chi_l(\varphi), \varphi \in (\varphi^*_{\chi,\mathbb{O}}, \pi/2)\}, \tag{5.13}$$

which connects $\tilde{v}_l^*$ to the spherical harmonic function of degree l which is axisymmetric with respect to the y-axis. For simplicity we combine the last two connections to

$$\Upsilon_l^{\chi,\mathbb{O}} := \{\chi_l(\varphi), \varphi \in (0, \pi/2)\}. \tag{5.14}$$

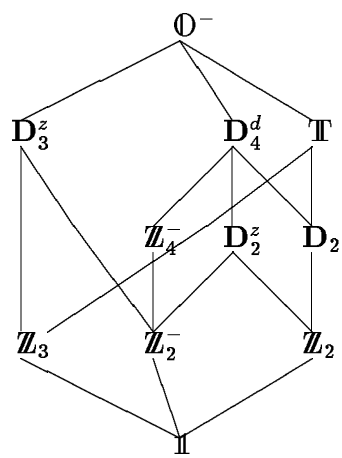
All other connections lie on the group orbit of $\Upsilon_l^{\omega,\mathbb{O}}$, $\Upsilon_l^{\chi,\mathbb{O},a}$ or $\Upsilon_l^{\chi,\mathbb{O},b}$.

Figure 5: $Fix_{(\mathbb{O}\oplus\mathbb{Z}_2^c, \mathbf{O}(3)/(\mathbf{O}(2)\oplus\mathbb{Z}_2^c))}$ and $Fix_{(\mathbb{O}^-, \mathbf{O}(3)/(\mathbf{O}(2)\oplus\mathbb{Z}_2^c))}$



Considering $L = \mathbb{O}\oplus\mathbb{Z}_2^c$ we first need the subgroups of $\mathbb{O}\oplus\mathbb{Z}_2^c$. Some are again of the form L' or $L'\oplus\mathbb{Z}_2^c$ where L' is a subgroup of $\mathbb{O}$. There is also a bunch of class III subgroups of $\mathbb{O}\oplus\mathbb{Z}_2^c$, but they are not relevant for the action on $\mathbf{O}(3)/(\mathbf{O}(2)\oplus\mathbb{Z}_2^c)$, because any element is clearly invariant under $\gamma = -\mathbb{1}$. Therefore all occurring stabilizers are of the form $L'\oplus\mathbb{Z}_2^c$. Compared with $L = \mathbb{O}$ we obtain the same fixed-point subspaces; just the stabilizers increase by $\gamma = -\mathbb{1}$ (cf. Figure 5).

Figure 6: Subgroups of $\mathbb{O}^-$



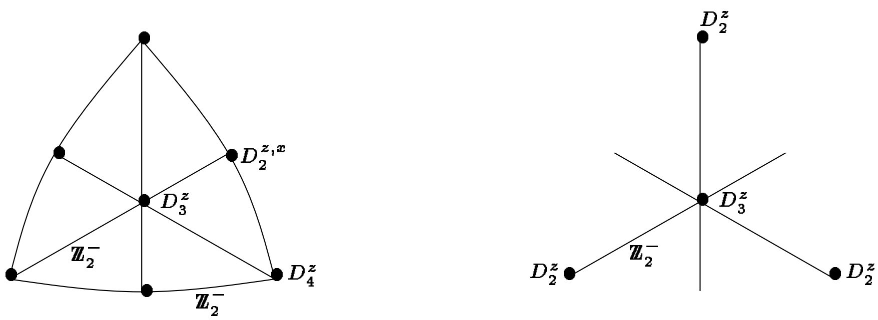
To discuss $L = \mathbb{O}^-$, we first give the subgroups of $\mathbb{O}^-$ in Figure 6. (cf. [5], Chapter XIII Proposition 9.4).

As to our usage of notation for the class III subgroups see again [5], Chapter XIII Theorem 7.5 (for instance $D_4^d = D_2 \cup \{-\gamma, \gamma \in D_4 \backslash D_2\}$). We have

$$
\begin{aligned}
Fix_{\mathbf{O}(3)/(\mathbf{O}(2)\oplus\mathbb{Z}_2^c)}(\mathbb{Z}_2) &\cong S^1 \cup 1pt, & Fix_{\mathbf{O}(3)/(\mathbf{O}(2)\oplus\mathbb{Z}_2^c)}(\mathbb{Z}_2^-) &\cong S^1 \cup 1pt \\
Fix_{\mathbf{O}(3)/(\mathbf{O}(2)\oplus\mathbb{Z}_2^c)}(D_3^z) &\cong 1pt, & Fix_{\mathbf{O}(3)/(\mathbf{O}(2)\oplus\mathbb{Z}_2^c)}(D_4^d) &\cong 1pt \\
Fix_{\mathbf{O}(3)/(\mathbf{O}(2)\oplus\mathbb{Z}_2^c)}(D_2^z) &\cong 3pt.
\end{aligned}
$$

All connections of $Fix_{(\mathbb{O}\oplus\mathbb{Z}_2^c,\mathbf{O}(3)/(\mathbf{O}(2)\oplus\mathbb{Z}_2^c))}$ and $Fix_{(\mathbb{O}^-,\mathbf{O}(3)/(\mathbf{O}(2)\oplus\mathbb{Z}_2^c))}$ have already been parametrized (by ω and χ) (cf. Figure 5).

Figure 7: $Fix_{(\mathbb{O}\oplus\mathbb{Z}_2^c,\mathbf{O}(3)/\mathbf{O}(2)^-)}$ and $Fix_{(\mathbb{O}^-,\mathbf{O}(3)/\mathbf{O}(2)^-)}$



It remains to discuss the case $H = \mathbf{O}(2)^-$. Since $\mathbf{O}(3)/\mathbf{O}(2)^- \cong S^2$ we have for $L = \mathbb{O}\oplus\mathbb{Z}_2^c$ as nontrivial fixed-point subspaces ($D_2^{z,x}$ denotes a D_2^z subgroup of $\mathbb{O}\oplus\mathbb{Z}_2^c$ with $D_2^z \not\subset D_4^z$)

$$Fix_{\mathbf{O}(3)/\mathbf{O}(2)^-}(\mathbb{Z}_2^-) \cong S^1, \qquad Fix_{\mathbf{O}(3)/\mathbf{O}(2)^-}(D_2^{z,x}) \cong 2pt$$
$$Fix_{\mathbf{O}(3)/\mathbf{O}(2)^-}(D_3^z) \cong 2pt, \qquad Fix_{\mathbf{O}(3)/\mathbf{O}(2)^-}(D_4^z) \cong 2pt.$$

$Fix_{(\mathbb{O}\oplus\mathbb{Z}_2^c,\mathbf{O}(3)/\mathbf{O}(2)^-)}$ is given in Figure 7. For $L = \mathbb{O}^-$ the stabilizers decrease. Since $D_4^z \not\subset \mathbb{O}^-$ these equilibria have only $D_2^z \subset D_4^z$ symmetry and since $D_2^{z,x} \not\subset \mathbb{O}^-$ these equilibria are now missing (cf. Figure 7). In the last case $L = \mathbb{O}$ no connection is left, because $\mathbb{Z}_2^- \not\subset \mathbb{O}$. The parametrizations for these $H = \mathbf{O}(2)^-$ cases, which we will need in the sequel, have been developed earlier (see ω and χ for the case $l = 3$).

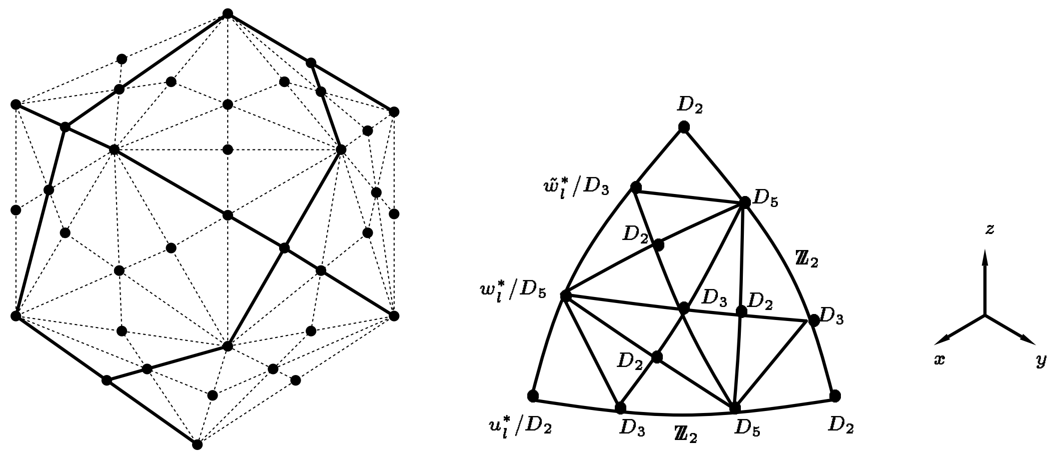
5.3 The Fixed-Point Subspaces in case $L = \mathbb{I}$, $\mathbb{I}\oplus\mathbb{Z}_2^c$ and $H = \mathbf{O}(2)\oplus\mathbb{Z}_2^c$, $\mathbf{O}(2)^-$

We begin once more with $G/H = \mathbf{O}(3)/(\mathbf{O}(2)\oplus\mathbb{Z}_2^c) \cong \mathbb{P}^2 \cong \mathbf{SO}(3)/\mathbf{O}(2)$ and let $L = \mathbb{I}$. Following [12], Table 1, we have

$$Fix_{\mathbf{O}(3)/(\mathbf{O}(2)\oplus\mathbb{Z}_2^c)}(\mathbb{Z}_2) \cong S^1 \cup 1pt, \qquad Fix_{\mathbf{O}(3)/(\mathbf{O}(2)\oplus\mathbb{Z}_2^c)}(D_2) \cong 3pt$$
$$Fix_{\mathbf{O}(3)/(\mathbf{O}(2)\oplus\mathbb{Z}_2^c)}(D_3) \cong 1pt, \qquad Fix_{\mathbf{O}(3)/(\mathbf{O}(2)\oplus\mathbb{Z}_2^c)}(D_5) \cong 1pt.$$

Here we get as in the octahedral case three independent different branches connecting equilibria. However, the parametrization is less difficult, since all of them can be found as subbranches of $\Upsilon_l^{\omega,\mathbb{T}}$. We set using $\varphi^*_{\omega,\mathbb{I},1} := \frac{1}{2}\arccos(\frac{\sqrt{5}}{5})$ and $\varphi^*_{\omega,\mathbb{I},2} := \frac{\pi}{2} - \arcsin(\frac{\sqrt{3}(-1+\sqrt{5})}{6})$

Figure 8: $Fix_{(\mathbb{I},\mathbf{O}(3)/(\mathbf{O}(2)\oplus\mathbb{Z}_2^c))}$



$$\begin{aligned}
\Upsilon_l^{\omega,\mathbb{I},a} &:= \{\omega_l(\varphi), \varphi \in (0, \varphi^*_{\omega,\mathbb{I},1})\} \\
\Upsilon_l^{\omega,\mathbb{I},b} &:= \{\omega_l(\varphi), \varphi \in (\varphi^*_{\omega,\mathbb{I},1}, \varphi^*_{\omega,\mathbb{I},2})\} \\
\Upsilon_l^{\omega,\mathbb{I},c} &:= \{\omega_l(\varphi), \varphi \in (\varphi^*_{\omega,\mathbb{I},2}, \pi/2\}.
\end{aligned}$$

For simplicity we combine them to

$$\Upsilon_l^{\omega,\mathbb{I}} := \{\omega_l(\varphi), \varphi \in (0, \pi/2)\} = \Upsilon_l^{\omega,\mathbb{T}}. \tag{5.15}$$

$\Upsilon_l^{\omega,\mathbb{I},a}$ connects an equilibrium in $Fix_{\mathbf{O}(3)/(\mathbf{O}(2)\oplus\mathbb{Z}_2^c)}(D_2)$ with one in $Fix_{\mathbf{O}(3)/(\mathbf{O}(2)\oplus\mathbb{Z}_2^c)}(D_5)$, i.e. for $l = 2$ it connects $2\,x^2 - (y^2 + z^2)$ with

$$w_2^* := \left(\frac{1}{2} + \frac{3}{10}\sqrt{5}\right) x^2 + \frac{6}{5}\sqrt{5}xz - y^2 + \left(\frac{1}{2} - \frac{3}{10}\sqrt{5}\right) z^2.$$

Similarly taking $l = 3$ gives a connection from $2\,x^3 - 3\,x(y^2 + z^2)$ to

$$\begin{aligned}
w_3^* \;:=\; & \frac{\sqrt{10}}{10}\sqrt{5-\sqrt{5}}x^3 - \frac{\sqrt{10}}{10}\sqrt{5+\sqrt{5}}z^3 + \frac{3}{5}\sqrt{5}\sqrt{5+2\sqrt{5}}x^2 z \\
& + \frac{3}{5}\sqrt{5}\sqrt{5-2\sqrt{5}}xz^2 - \frac{3}{10}\sqrt{10}\sqrt{5-\sqrt{5}}y^2 z - \frac{3\sqrt{10}}{10}\sqrt{5+\sqrt{5}}y^2 x.
\end{aligned}$$

$\Upsilon_l^{\omega,\mathbb{I},b}$ connects an equilibrium in $Fix_{\mathbf{O}(3)/(\mathbf{O}(2)\oplus\mathbb{Z}_2^c)}(D_5)$ with one in $Fix_{\mathbf{O}(3)/(\mathbf{O}(2)\oplus\mathbb{Z}_2^c)}(D_3)$. The equilibrium in $Fix_{\mathbf{O}(3)/(\mathbf{O}(2)\oplus\mathbb{Z}_2^c)}(D_3)$ for $l=2$ is

$$\tilde{w}_2^* := \frac{1}{2}\left(1-\sqrt{5}\right)x^2 + \frac{1}{2}\left(1+\sqrt{5}\right)z^2 - y^2 + 2\,xz.$$

For $l=3$ we obtain

$$\begin{aligned}\tilde{w}_3^* \;:=\; & -\frac{\sqrt{3}}{36}\left(2\,\left(11-\sqrt{5}\right)x^3 - 2\,\left(11+\sqrt{5}\right)z^3 + 12\,\left(4-\sqrt{5}\right)x^2 z\right.\\ & \left.-12\,\left(4+\sqrt{5}\right)xz^2 - 18\,\left(+1-\sqrt{5}\right)xy^2 + 18\,\left(1+\sqrt{5}\right)zy^2\right).\end{aligned}$$

At last $\Upsilon_l^{\omega,\mathbb{I},c}$ connect these equilibria again with $Fix_{\mathbf{O}(3)/(\mathbf{O}(2)\oplus\mathbb{Z}_2^c)}(D_2) : 2\,z^2 - (x^2+y^2)$ in the case $l=2$ and $2\,z^3 - 3\,z(x^2+y^2)$ in the case $l=3$.

The discussion of $L = \mathbb{I}\oplus\mathbb{Z}_2^c$ and $H = \mathbf{O}(2)\oplus\mathbb{Z}_2^c$ gives again only the additional symmetry $\gamma = -\mathbb{1}$. The remaining cases $L = \mathbb{I}, \mathbb{I}\oplus\mathbb{Z}_2^c$ and $H = \mathbf{O}(2)^-$ can be discussed as in the tetrahedral case. In any case, connections which occur are already parameterized.

6 Basic Flows for Perturbations of the Reaction Term

The aim of this section is to calculate the direction of the flow on connections $\Upsilon \in \mathcal{H}_{(L,G/H)}$ in the case $G = \mathbf{O}(3)$, $H = \mathbf{O}(2)\oplus\mathbb{Z}_2^c$ or $H = \mathbf{O}(2)^-$ and for L a supergroup of $\mathbb{T}$. We firstly perform a case study using perturbations of the reaction term for (1.8) of the form $h : D \subset L^2(S^2) \to L^2(S^2)$,

$$h(u)(x) = \bar{p}(x)\cdot\Theta(u(x)), \quad x \in S^2, \tag{6.1}$$

where $\bar{p} \in \bar{\mathcal{R}}^L$ and $\Theta : \mathbb{R} \to \mathbb{R}$ is a smooth function. Following Theorem 2.2 on the connection $\Upsilon := \{\omega(\varphi) \mid \varphi \in (0,\varphi^*)\}$ we have to calculate

$$\mathcal{F}_\Upsilon^{(\bar{p},\Theta)}(\varphi) := \int_{S^2} \mathfrak{T}(\varphi)\cdot\bar{p}\cdot\Theta(\omega(\varphi))dS, \quad \varphi \in [0,\varphi^*], \tag{6.2}$$

where we use the tangent vector $\mathfrak{T}(\varphi) := \frac{d}{d\varphi}\omega(\varphi)$ without normalization (see Remark 2.22). By a 'basic flow' we mean a function $\mathcal{G} : [0,\varphi^*] \to \mathbb{R}$ which is achieved in (6.2) by a specific choice of $\bar{p}, \Theta$ and Υ. We speak of basic flows, although (6.2) actually just gives the direction of the flow. Note that by construction $\mathcal{G}(0) = \mathcal{G}(\varphi^*) = 0$, because the endpoints of every connection are equilibria of $(L, G/H)$ (cf. (1.12)). For simplicity, we restrict ourselves to the case $\Theta(\omega) = k\omega^{k-1}$, $k \in \mathbb{N}$. Here, we use

$$\mathcal{F}_{\Upsilon}^{(\bar{p},k)}(\varphi) := \int_{S^2} \mathfrak{T}(\varphi) \cdot \bar{p} \cdot k\omega(\varphi)^{k-1} dS = \frac{d}{d\varphi} \int_{S^2} \bar{p} \cdot \omega(\varphi)^k dS = \frac{d}{d\varphi} (\bar{p}, \omega(\varphi)^k)_{L^2(S^2)}. \quad (6.3)$$

To obtain the parametrizations of the connections Υ in Section 5, we had to assume that the kernel $\ker A(\lambda_0)$ is an irreducible representation of $\mathbf{O}(3)$. This gives by Lemma 5.1 that

$$\ker A(\lambda_0) = \mathcal{SH}_l, \text{ for some } l \in \mathbb{N}_0. \quad (6.4)$$

We will explicitly calculate the basic flows for some $\bar{p}$ of low degree, as well as for some small l. Our goal is to understand the basic flows which occur for $\bar{p} \in \bar{\mathcal{R}}^{\mathbb{T}}$. According to Theorem 4.6 it is sufficient to discuss $\bar{p} \in \bar{\mathcal{R}}^{\mathbb{O}\oplus\mathbb{Z}_2^c}$, $\bar{\tau}_3 \cdot \bar{\mathcal{R}}^{\mathbb{O}\oplus\mathbb{Z}_2^c}$, $\bar{\tau}_3\bar{\tau}_6 \cdot \bar{\mathcal{R}}^{\mathbb{O}\oplus\mathbb{Z}_2^c}$, and $\bar{\tau}_6 \cdot \bar{\mathcal{R}}^{\mathbb{O}\oplus\mathbb{Z}_2^c}$ separately. We remark again that $\bar{\tau}_3 \cdot \bar{\mathcal{R}}^{\mathbb{O}\oplus\mathbb{Z}_2^c} \subset \bar{\mathcal{R}}^{\mathbb{O}^-}$ gives all precisely $\mathbb{O}^-$–invariant polynomials and the polynomials of the form $\bar{\tau}_3\bar{\tau}_6 \cdot \bar{\mathcal{R}}^{\mathbb{O}\oplus\mathbb{Z}_2^c} \subset \bar{\mathcal{R}}^{\mathbb{O}}$ are precisely $\mathbb{O}$–invariant (cf. Theorem 4.7).

Polynomials in $\bar{\tau}_6 \cdot \bar{\mathcal{R}}^{\mathbb{O}\oplus\mathbb{Z}_2^c}$ are precisely $\mathbb{T}\oplus\mathbb{Z}_2^c$–invariant. Nevertheless, some of them can be written as a sum of $\mathbb{O}\oplus\mathbb{Z}_2^c$– and $\mathbb{I}\oplus\mathbb{Z}_2^c$–invariant polynomials (cf. Theorem 4.8). The best chance to see tetrahedral flows, which are not influenced by any additional symmetry is to use $\bar{p} \in W^{\bar{\mathcal{R}}}$. The basic flows obtained in any of the above cases might then (to some extend) be used to generate new $\mathbb{T}$–equivariant flows by linear combination. One only has to ensure to combine flows obtained for the same k (in order to have homogeneous perturbations h – see (2.17)). Furthermore the combined flow has to have only simple zeros to make Theorem 2.2 applicable.

6.1 Basic Flows for $L = \mathbb{O}\oplus\mathbb{Z}_2^c$ Symmetry

For both $H = \mathbf{O}(2)\oplus\mathbb{Z}_2^c$ and $\mathbf{O}(2)^-$ there are basically two different parametrizations for three connections of $(\mathbb{O}\oplus\mathbb{Z}_2^c, \mathbf{O}(3)/H)$ which we have to discuss (cf. Subsection 5.2): $\Upsilon_l^{\omega,\mathbb{O}}$ and $\Upsilon_l^{\chi,\mathbb{O}}$. We simplify notation, setting

$$\mathcal{F}_{l,\omega}^{(\bar{p},k)}(\varphi) := \mathcal{F}_{\Upsilon_l^{\omega,\mathbb{O}}}^{(\bar{p},k)}(\varphi), \quad \varphi \in [0, \varphi_{\omega,\mathbb{O}}^* = \pi/4] \quad (6.5)$$

and analogously for $\Upsilon_l^{\chi,\mathbb{O}}$. At next we give a sample calculation for some specific (overviewable) data; we calculate the flows for $k = l = 2$ and $\bar{p} = \bar{\rho}_4$. We have

$$\mathcal{F}_{2,\omega}^{(\bar{\rho}_4,2)}(\varphi) = \frac{d}{d\varphi} (\bar{\rho}_4, \omega_2(\varphi)^2)_{L^2(S^2)}.$$

Let $b_{xx} := 2 - 3\sin^2(\varphi) = -1 + 3\cos^2(\varphi)$, $b_{zz} := 2 - 3\cos^2(\varphi)$ and $b_{xz} := 6\sin(\varphi)\cos(\varphi)$. We obtain from (5.6)

$$\begin{aligned}
\bar\rho_4 \cdot \omega_2(\varphi)^2 \;=\;& \left(2\, b_{xx}\, b_{zz} + b_{xz}^2\right) x^2y^4z^2 - 2\, b_{zz} x^4y^2z^2 - 2\, b_{xx}\, x^2y^2z^4 \\
&+ \left(1 + b_{xx}^2\right) x^4y^4 + \left(b_{xx}^2 + b_{zz}^2\right) x^4z^4 + \left(1 + b_{zz}^2\right) y^4z^4 \\
&+ \left(2\, b_{xx}\, b_{zz} + b_{xz}^2\right) (x^6z^2 + x^2z^6) - 2\, b_{xx}\, (x^6y^2 + x^2y^6) \\
&-2\, b_{zz}\, (z^2y^6 + y^2z^6) + b_{xx}^2\, x^8 + y^8 + b_{zz}^2\, z^8 \\
&+2\, b_{xx}\, b_{xz}\, (x^7z + x^3y^4z + x^3z^5) + 2\, b_{zz}\, b_{xz}\, (x^5z^3 + xy^4z^3 + xz^7) \\
&-2\, b_{xz}\, (x^5y^2z + xy^6z + xy^2z^5).
\end{aligned}$$

Using

$$\int_{S^2} x^i\, y^j\, z^m dS = \int_{S^2} x^{\sigma(i)}\, y^{\sigma(j)}\, z^{\sigma(m)} dS \tag{6.6}$$

for any permutation σ of (i, j, m) and

$$\int_{S^2} x^i\, y^j\, z^m dS = 0 \text{ for } i, j, m \in \mathbb{N}_0 \text{ and } i, j \text{ or } m \text{ odd }, \tag{6.7}$$

we derive

$$\begin{aligned}
(\bar\rho_4, \omega_2(\varphi)^2)_{L^2(S^2)} &= \left(1 + b_{xx}^2 + b_{zz}^2\right) \left(2 \int_{S^2} y^4\, z^4 dS + \int_{S^2} z^8 dS\right) \\
&+ \left(2\, b_{xx}\, b_{zz} - 2\,(b_{zz} + b_{xx}) + b_{xz}^2\right) \left(\int_{S^2} x^2\, y^2\, z^4 dS + 2 \int_{S^2} y^2\, z^6 dS\right).
\end{aligned}$$

All these elementary integrals over S^2 can be easily calculated (cf. Section 9):

$$\int_{S^2} x^2\, y^2\, z^4 dS = \frac{4}{315}\,\pi, \qquad \int_{S^2} y^2\, z^6 dS = \frac{4}{63}\,\pi$$
$$\int_{S^2} y^4\, z^4 dS = \frac{4}{105}\,\pi, \qquad \int_{S^2} z^8 dS = \frac{4}{9}\,\pi.$$

We conclude

$$\begin{aligned}
(\bar\rho_4, \omega_2(\varphi)^2)_{L^2(S^2)} &= \frac{88}{315}\,\pi \left(b_{xx}\, b_{zz} - b_{xx} - b_{zz} + \frac{1}{2} b_{xz}^2\right) + \frac{164}{315}\,\pi \left(1 + b_{xx}^2 + b_{zz}^2\right) \\
&= \frac{16}{7}\,\pi - \frac{64}{35}\,\pi\, \cos^2(\varphi) + \frac{64}{35}\,\pi\, \cos^4(\varphi),
\end{aligned}$$

and after differentiating

$$\mathcal{F}_{2,\omega}^{(\bar{\rho}_4,2)}(\varphi) = \frac{128}{35}\pi\ \cos(\varphi)\sin(\varphi)\left(1 - 2\ \cos^2(\varphi)\right).$$

In the same manner, using (5.11), we calculate straight forward

$$\mathcal{F}_{2,\chi}^{(\bar{\rho}_4,2)}(\varphi) = \frac{64}{35}\pi\ \cos(\varphi)\sin(\varphi)\left(2 - 3\ \cos^2(\varphi)\right).$$

In the sequel, we do not give any further details on such calculations, since they all can conveniently be done by any symbolic calculation program (see Section 9 for more details). The former example gives us the first basic flow. Using

$$\kappa_{ij}(\varphi) := i - j \cdot \cos^2(\varphi) \text{ and } \eta(\varphi) := \cos(\varphi)\sin(\varphi)$$

we define for $\bar{\varphi} := (\varphi_\omega, \varphi_\chi) \in [0, \varphi^*_{\omega,\mathbb{O}} = \pi/4] \times [0, \pi/2]$

$$\mathcal{G}_1^{\mathbb{O}\oplus\mathbb{Z}_2^c}(\bar{\varphi}) := \left(\mathcal{G}_{1,\omega}^{\mathbb{O}\oplus\mathbb{Z}_2^c}(\varphi_\omega), \mathcal{G}_{1,\chi}^{\mathbb{O}\oplus\mathbb{Z}_2^c}(\varphi_\chi)\right) := \left((-2)\cdot\kappa_{12}(\varphi_\omega)\eta(\varphi_\omega), -\kappa_{23}(\varphi_\chi)\eta(\varphi_\chi)\right).$$

Collecting the flows on ω and χ to

$$\mathcal{F}_l^{(\bar{p},k)}(\bar{\varphi}) := \left(\mathcal{F}_{l,\omega}^{(\bar{p},k)}(\varphi_\omega), \mathcal{F}_{l,\chi}^{(\bar{p},k)}(\varphi_\chi)\right)$$

we have proved

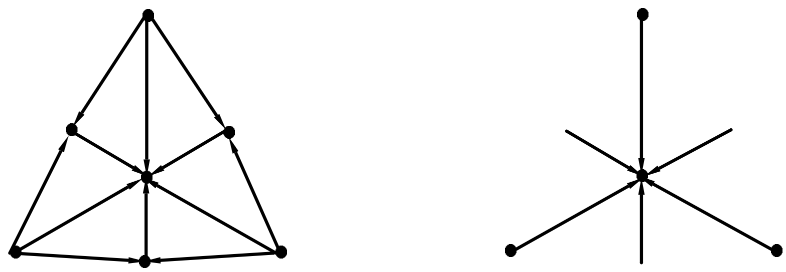
Theorem 6.1 *The flow (direction) for $k = l = 2$ and $\bar{p} = \bar{\rho}_4$ is given by*

$$\mathcal{F}_2^{(\bar{\rho}_4,2)} = -\frac{64}{35}\pi \cdot \mathcal{G}_1^{\mathbb{O}\oplus\mathbb{Z}_2^c}.$$

Thus, under the assumptions of Theorem 2.2 (with $L = \mathbb{O}\oplus\mathbb{Z}_2^c$ and $H = \mathbf{O}(2)\oplus\mathbb{Z}_2^c$) for the $l = 2$ representation on $\ker A(\lambda_0)$, we find for the semilinear parabolic equation (1.8) with perturbation (6.1), $\bar{p} = \bar{\rho}_4$ and $\Theta(\omega) = 2\omega$, heteroclinic orbits for the perturbed flow. $\mathcal{G}_1^{\mathbb{O}\oplus\mathbb{Z}_2^c}$ is illustrated in the left diagram of Figure 9.

Remark 6.2 *This kind of flow occurs actually quite frequently (up to a multiple). It is also achieved for instance by the following perturbations $(\bar{p}; k)$: $(\bar{\rho}_4; k = 3,4,5,6)$, $(\bar{\rho}_6; 2)$, $(\bar{\rho}_4^2; 2)$, $(\bar{\rho}_6^2; 2)$, $(\bar{\rho}_4\bar{\rho}_6; 2)$, $(\bar{\rho}_4^3; 2)$, $(\bar{\rho}_6^3; 2)$, $(\bar{\rho}_4^2\bar{\rho}_6; 2)$, $(\bar{\rho}_4\bar{\rho}_6^2; 2)$ for $l = 2$ and $(\bar{\rho}_4; k = 2,4)$ for $l = 3$.*

Figure 9: $\mathcal{G}_1^{\mathbb{O}\oplus\mathbb{Z}_2^c}$ and $\mathcal{G}_1^{\mathbb{O}^-}$



There are much more basic flows with $\mathbb{O}\oplus\mathbb{Z}_2^c$ symmetry, like for instance $\mathcal{G}_2^{\mathbb{O}\oplus\mathbb{Z}_2^c}(\bar{\varphi}) := (0, \cos^2(\varphi_\chi)) * \mathcal{G}_1^{\mathbb{O}\oplus\mathbb{Z}_2^c}(\bar{\varphi})$ or $\mathcal{G}_3^{\mathbb{O}\oplus\mathbb{Z}_2^c}(\bar{\varphi}) := \Big((-8)\ \sin^2(\varphi_\omega)\cos^2(\varphi_\omega), 3\ \cos^4(\varphi_\chi)\Big) * \mathcal{G}_1^{\mathbb{O}\oplus\mathbb{Z}_2^c}(\bar{\varphi})$ (here the product '*' of two vectors is the product in each component). However, in both of these cases Theorem 2.2 is not applicable, since the zeros are not simple. Therefore we do not pursue this any further, although perturbations generating these basic flows may very well be treated together with the perturbations yielding $\mathcal{G}_1^{\mathbb{O}\oplus\mathbb{Z}_2^c}$, as long as the latter are dominant (which happens e.g. for $(\bar{\rho}_6; k = 3, l = 2)$ and $(\bar{\rho}_4\bar{\rho}_6; k = 4, l = 2)$).

In order to see a heteroclinic cycle in the case $L = \mathbb{O}\oplus\mathbb{Z}_2^c$, the flow along $\Upsilon_l^{\chi,\mathbb{O}}$ should have no sign change. In that case at the fixed-point in the middle ($\varphi = \varphi^*_{\chi,\mathbb{O}} = \arccos(\frac{\sqrt{6}}{3})$) a double zero of $\mathcal{F}_{l,\chi}^{(\bar{p},k)}$ had to occur. This is not only a situation which Theorem 2.2 could not handle, but furthermore, the D_3 fixed-point in the middle would be a degenerate fixed-point for the flow (yielding a non hyperbolic equilibrium), which is not a generic situation.

6.2 Basic Flows for $L = \mathbb{O}$ Symmetry

In the case $H = \mathbf{O}(2)^-$ (this corresponds to irreducible representations of $\ker A(\lambda_0)$ with l odd) we have found in Section 5.2 that $Fix_{(\mathbb{O},\mathbf{O}(3)/\mathbf{O}(2)^-)}$ contains only isolated points. Theorem 2.2 is not applicable, since there are no connections of $(\mathbb{O}, \mathbf{O}(3)/\mathbf{O}(2)^-)$.

In the case $H = \mathbf{O}(2)\oplus\mathbb{Z}_2^c$ (i.e. l even), however, $Fix_{(\mathbb{O},\mathbf{O}(3)/(\mathbf{O}(2)\oplus\mathbb{Z}_2^c))}$ contains the same connections as $Fix_{(\mathbb{O}\oplus\mathbb{Z}_2^c,\mathbf{O}(3)/(\mathbf{O}(2)\oplus\mathbb{Z}_2^c))}$. By Theorem 4.7 the polynomials with precisely $\mathbb{O}$ symmetry are $\bar{\tau}_3\bar{\tau}_6\bar{\mathcal{R}}^{\mathbb{O}\oplus\mathbb{Z}_2^c}$. These perturbations give just the trivial flow (which means Theorem 2.2 is again not applicable):

Theorem 6.3 *For all irreducible representations of* $\mathbf{O}(3)$ *on* $\ker A(\lambda_0)$ *and with* $H = \mathbf{O}(2)\oplus\mathbb{Z}_2^c$ *(which corresponds to even* l*) we obtain for all perturbations* $\bar{p} \in \bar{\tau}_3\bar{\tau}_6\bar{\mathcal{R}}^{\mathbb{O}\oplus\mathbb{Z}_2^c} \subset \bar{\mathcal{R}}^{\mathbb{O}}$ *just the trivial flow* ($k \in \mathbb{N}$)*:*

$$\mathcal{F}_l^{(\bar{p},k)}(\bar{\varphi}) = \left(\mathcal{F}_{l,\omega}^{(\bar{p},k)}(\varphi_\omega), \mathcal{F}_{l,\chi}^{(\bar{p},k)}(\varphi_\chi)\right) \equiv (0,0).$$

Proof. Consider for instance

$$\mathcal{F}_{l,\chi}^{(\bar{p},k)}(\varphi) = \frac{d}{d\varphi}(\bar{p}, \chi_l(\varphi)^k)_{L^2(S^2)}, \quad \varphi \in [0, \pi/2].$$

$\chi_l(\varphi)$ is a sum of homogeneous polynomials of degree l. Hence $\chi_l(\varphi)^k$ is a sum of homogeneous polynomials of degree $k \cdot l$ and since l is even, so is $k \cdot l$. On the other hand, $\bar{p} \in \bar{\tau}_3\bar{\tau}_6\bar{\mathcal{R}}^{\mathbb{O}\oplus\mathbb{Z}_2^c}$ is a sum of homogeneous polynomials of odd degree, since the generators of $\bar{\mathcal{R}}^{\mathbb{O}\oplus\mathbb{Z}_2^c}$ have only even degree. Altogether, $\bar{p} \cdot \chi_l(\varphi)^k$ is a sum of homogeneous polynomials of odd degree. However, integration of homogeneous polynomials of odd degree yields 0 (cf. (6.7)) and the proof is established. □

6.3 Basic Flows for $L = \mathbb{O}^-$ Symmetry

From Subsection 5.2 we know that $Fix_{(\mathbb{O}^-, \mathbf{O}(3)/(\mathbf{O}(2)\oplus\mathbb{Z}_2^c))}$ contains connections which are parametrized by ω and χ, whereas the relevant connections in $Fix_{(\mathbb{O}^-, \mathbf{O}(3)/\mathbf{O}(2)^-)}$ are given by χ only. The $\mathbb{O}^-$ perturbations of interest are of the form $\bar{\tau}_3\bar{\mathcal{R}}^{\mathbb{O}\oplus\mathbb{Z}_2^c}$. Hence, for $H = \mathbf{O}(2)\oplus\mathbb{Z}_2^c$ we have for the same reason as in Theorem 6.3:

Theorem 6.4 *For all irreducible representations of* $\mathbf{O}(3)$ *on* $\ker A(\lambda_0)$ *and with* $H = \mathbf{O}(2)\oplus\mathbb{Z}_2^c$ *(i.e.* l *even) any perturbation* $\bar{p} \in \bar{\tau}_3\bar{\mathcal{R}}^{\mathbb{O}\oplus\mathbb{Z}_2^c} \subset \bar{\mathcal{R}}^{\mathbb{O}^-}$ *gives just the trivial flow* $(k \in \mathbb{N})$*:*

$$\mathcal{F}_l^{(\bar{p},k)}(\bar{\varphi}) = \left(\mathcal{F}_{l,\omega}^{(\bar{p},k)}(\varphi_\omega), \mathcal{F}_{l,\chi}^{(\bar{p},k)}(\varphi_\chi)\right) \equiv (0,0).$$

For $H = \mathbf{O}(2)^-$ (and l odd) we just have to consider the connection χ. Note that the connection of $(\mathbb{O}^-, \mathbf{O}(3)/\mathbf{O}(2)^-)$ which connects two D_3^z equilibria is only half parametrized by χ (cf. Figure 7). This, however, does not matter, since the flow on the other part is obtained by a reflection. Similar to Theorem 6.4 we have for even k the trivial flow:

Theorem 6.5 *For all irreducible representations of* $\mathbf{O}(3)$ *on* $\ker A(\lambda_0)$ *and with* $H = \mathbf{O}(2)^-$ *(i.e.* l *odd) we get for all perturbations* $\bar{p} \in \bar{\tau}_3\bar{\mathcal{R}}^{\mathbb{O}\oplus\mathbb{Z}_2^c} \subset \bar{\mathcal{R}}^{\mathbb{O}^-}$ *and for even* $k \in \mathbb{N}$ *just the trivial flow:*

$$\mathcal{F}_{l,\chi}^{(\bar{p},k)}(\varphi) \equiv 0, \quad \varphi \in [0, \pi/2].$$

Proof. The proof is done along the lines of the proof of Theorem 6.3. □

Therefore only for odd k and odd l $\mathbb{O}^-$−perturbations might yield situations, where Theorem 2.2 is applicable. Some of them indeed do.

Theorem 6.6 *The flow (direction) in case $k = 1, l = 3$ and $\bar{p} = \bar{\tau}_3$ is given by*

$$\mathcal{F}_{3,\chi}^{(\bar{\tau}_3,1)}(\varphi) = -\frac{4}{7}\pi\kappa_{23}(\varphi)\cos(\varphi) =: \frac{4}{7}\pi\mathcal{G}_1^{\mathbb{O}^-}(\varphi), \quad \varphi \in [0, \frac{\pi}{2}]. \tag{6.8}$$

Thus, in case the assumptions of Theorem 2.2 ($L = \mathbb{O}^-$ and $H = \mathbf{O}(2)^-$) are satisfied for the $l = 3$ representation on $\ker A(\lambda_0)$*, we find for the semilinear parabolic equation (1.8) with perturbation (6.1), $\bar{p} = \bar{\tau}_3$ and $\Theta(\omega) = 1$, heteroclinic orbits. $\mathcal{G}_1^{\mathbb{O}^-}$ is shown in Figure 9.*

Remark 6.7 *Again this kind of flow occurs quite frequently (up to a multiple). It is achieved for instance by the following perturbations $(\bar{p}; k)$ and $l = 3$: $(\bar{\tau}_3; k = 3, 5, 7)$, $(\bar{\tau}_3\bar{\rho}_4; 1)$, $(\bar{\tau}_3\bar{\rho}_6; 1), (\bar{\tau}_3\bar{\rho}_4^2; 1), (\bar{\tau}_3\bar{\rho}_6^2; 1), (\bar{\tau}_3\bar{\rho}_4\bar{\rho}_6; 1)$.*

Other evaluations of the flow formula give e.g. $\mathcal{G}_2^{\mathbb{O}^-}(\varphi) := (\cos^2(\varphi)\cdot(7\cos^2(\varphi) - 8))\cdot \mathcal{G}_1^{\mathbb{O}^-}(\varphi)$ or $\mathcal{G}_3^{\mathbb{O}^-}(\varphi) := \left(\cos^4(\varphi)\sin^2(\varphi)\right)\cdot\mathcal{G}_1^{\mathbb{O}^-}(\varphi)$, but Theorem 2.2 does not apply, except, if flows do appear combined with $\mathcal{G}_1^{\mathbb{O}^-}$ and $\mathcal{G}_1^{\mathbb{O}^-}$ is dominant (use for instance $(\bar{\tau}_3\bar{\rho}_4; 3)$ and $(\bar{\tau}_3\bar{\rho}_6; 3)$ for $l = 3$).

6.4 Basic Flows for $L = \mathbb{I}\oplus\mathbb{Z}_2^c$ Symmetry

Here we have to consider (cf. Subsection 5.3)

$$\mathcal{F}_{l,\omega}^{(\bar{p},k)}(\varphi) = \frac{d}{d\varphi}(\bar{p}, \omega_l(\varphi)^k)_{L^2(S^2)}, \quad \varphi \in [0, \frac{\pi}{2}],$$

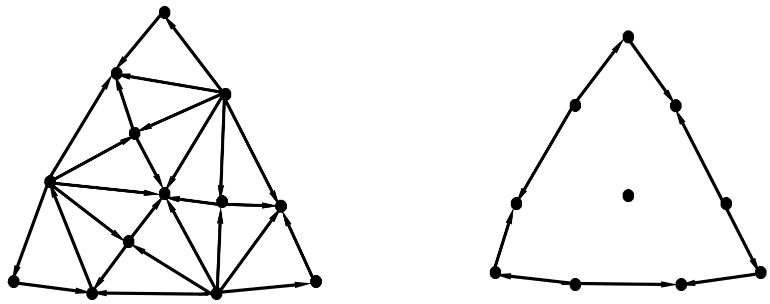
which parametrizes all three important connections at once.

Theorem 6.8 *The flow (direction) with $k = 3, l = 2$ and $\bar{p} = \bar{\iota}_6$ is given by*

$$\begin{aligned}
\mathcal{F}_{2,\omega}^{(\bar{\iota}_6,3)}(\varphi) &= -\frac{1152}{5005}\pi\,\sin(\varphi)\cos(\varphi)\left(5\left(1 - 6\sin^2(\varphi)\cos^2(\varphi)\right) - \sqrt{5}\kappa_{12}(\varphi)\right)\\
&=: \frac{1152}{5005}\pi\cdot\mathcal{G}_1^{\mathbb{I}\oplus\mathbb{Z}_2^c}(\varphi).
\end{aligned}$$

If the usual assumptions of Theorem 2.2 ($L = \mathbb{I}\oplus\mathbb{Z}_2^c$ and $H = \mathbf{O}(2)\oplus\mathbb{Z}_2^c$) are satisfied, we find heteroclinic orbits as in Figure 10.

Figure 10: $\mathcal{G}_1^{\mathbb{I}\oplus\mathbb{Z}_2^c}$ and $\mathcal{G}_1^{\mathbb{T}\oplus\mathbb{Z}_2^c}$



In the $\mathcal{G}_1^{\mathbb{I}\oplus\mathbb{Z}_2^c}$ picture, the D_5 equilibria are unstable and the D_3 equilibria are stable. Only the D_2 equilibria are hyperbolic.

Remark 6.9 *Other perturbations* $(\bar{p}; k)$ *which yield this flow (up to a multiple) are e.g.:* $(\bar{\iota}_6; k = 4, 5, 6), (\bar{\iota}_6^2; k = 3, 4)$ *for* $l = 2$ *and* $(\bar{\iota}_6; k = 2, 4), (\bar{\iota}_6^2; k = 2)$ *for* $l = 3$ *(i.e.* $H = \mathbf{O}(2)^-$*).*

Another basic flow which occurs is

$$\begin{aligned}\mathcal{G}_2^{\mathbb{I}\oplus\mathbb{Z}_2^c}(\varphi) \quad := \quad & \cos^3(\varphi)\,\sin^3(\varphi) \\ & \cdot \Big((1 - 5\,\cos^2(\varphi)\,\sin^2(\varphi)) + \sqrt{5}(1 - 6\,\sin^2(\varphi)\cos^2(\varphi))\kappa_{12}(\varphi)\Big)\,,\end{aligned}$$

but it contains nonsimple zeros. A sum of $\mathcal{G}_1^{\mathbb{I}\oplus\mathbb{Z}_2^c}$ and $\mathcal{G}_2^{\mathbb{I}\oplus\mathbb{Z}_2^c}$, where Theorem 2.2 can be applied, is achieved e.g. by $(\bar{\iota}_6^2; k = 5, l = 2)$. In this case, as for $L = \mathbb{O}\oplus\mathbb{Z}_2^c$, heteroclinic cycles cannot be generic, because both the D_3 and the D_5 fixed-point would be non hyperbolic saddles for the flow.

6.5 Basic Flows for $L = \mathbb{T}\oplus\mathbb{Z}_2^c$ Symmetry

Sums of $\mathbb{O}\oplus\mathbb{Z}_2^c-$ and $\mathbb{I}\oplus\mathbb{Z}_2^c-$Invariants

The only relevant connection in this case is $\omega = \omega_l(\varphi)$ for $\varphi \in [0, \frac{\pi}{2}]$. We are firstly going to consider perturbations $\bar{p} \in \bar{\mathcal{R}}^{\mathbb{T}\oplus\mathbb{Z}_2^c}$ which are sums of polynomials from $\bar{\mathcal{R}}^{\mathbb{O}\oplus\mathbb{Z}_2^c}$ and $\bar{\mathcal{R}}^{\mathbb{I}\oplus\mathbb{Z}_2^c}$. Due to the special structure of the forced zeros of flows related to $\mathbb{O}\oplus\mathbb{Z}_2^c$ and $\mathbb{I}\oplus\mathbb{Z}_2^c$ symmetry, we expect that the sum of these two flows contains not only the zeros which are forced by the group action.

Theorem 6.10

$$\begin{aligned}\mathcal{F}_{2,\omega}^{(\bar\tau_6,3)}(\varphi) &= -\frac{1152}{1001}\pi\,\Big(1-6\,\cos^2(\varphi)+6\,\cos^4(\varphi)\Big)\cos(\varphi)\sin(\varphi)\\ &=: -\frac{1152}{1001}\pi\,\mathcal{G}_1^{\mathbb{T}\oplus\mathbb{Z}_2^c}(\varphi).\end{aligned}$$

Hence, under the usual assumptions of Theorem 2.2 ($L = \mathbb{T}\oplus\mathbb{Z}_2^c$ and $H = \mathbf{O}(2)\oplus\mathbb{Z}_2^c$) for the $l = 2$ representation of $\ker A(\lambda_0)$, we find heteroclinic orbits for $\bar p = \bar\tau_6$ and $\Theta(\omega) = 3\omega^2$. $\mathcal{G}_1^{\mathbb{T}\oplus\mathbb{Z}_2^c}$ is illustrated in Figure 10.

Remark 6.11 *The same kind of flow is also achieved by perturbations $(\bar p; k)$ like: $(\bar\tau_6, k = 4,5,6), (\bar\tau_6\bar\rho_4; k = 3,4), (\bar\tau_6\bar\rho_6; k = 3,4), (\bar\tau_6\bar\rho_4^2; k = 3,4), (\bar\tau_6\bar\rho_4\bar\rho_6; k = 3,4), (\bar\tau_6\bar\rho_6^2; k = 3,4)$ for $l = 2$ and $(\bar\tau_6; k = 2,4), (\bar\tau_6\bar\rho_4; 2), (\bar\tau_6\bar\rho_6; 2), (\bar\tau_6\bar\rho_4^2; 2), (\bar\tau_6\bar\rho_6^2; 2), (\bar\tau_6\bar\rho_4\bar\rho_6; 2)$ for $l = 3$.*

We also observe $\mathcal{G}_2^{\mathbb{T}\oplus\mathbb{Z}_2^c}(\varphi) := \Big(1-5\,\cos^2(\varphi)\sin^2(\varphi)\Big)\sin^3(\varphi)\cos^3(\varphi)$ as an evaluation of the flow formula, but Theorem 2.2 is here not applicable. A sum of $\mathcal{G}_1^{\mathbb{T}\oplus\mathbb{Z}_2^c}$ and $\mathcal{G}_2^{\mathbb{T}\oplus\mathbb{Z}_2^c}$, where Theorem 2.2 still applies, appears e.g. for $(\bar\tau_6\bar\rho_4; k = 5, l = 2)$.

Invariants in $W^{\bar{\mathcal{R}}}$

Following our observations from above, the only chance left to find a heteroclinic cycle for perturbations of the reaction term is using $\bar p \in W^{\bar{\mathcal{R}}}$ (cf. Section 4.1). We obtain for instance

$$\begin{aligned}&\mathcal{F}_{2,\omega}^{(\bar w_{14}^{\bar{\mathcal{R}}},7)}(\varphi) =\\ &= \frac{294912}{26930125}\pi\,\sin^2(\varphi)\cos^2(\varphi)\,\kappa_{12}(\varphi)(1-5\,\cos^2(\varphi)\sin^2(\varphi))\,(1-9\,\cos^2(\varphi)\sin^2(\varphi)).\end{aligned}$$

To this flow not only Theorem 2.2 is not applicable, but furthermore it contains lots of additional zeros.

6.6 A Summary: Basic Flows for $L = \mathbb{T}$

Since any of the precedingly discussed groups for L have been supergroups of $\mathbb{T}$, we observe all these flows for $\mathbb{T}$ perturbations all well. Moreover, flows related to the same k might be added (as long as the zeros remain simple) to obtain new kinds of flows. Therefore, so far we were able to show the existence of many heteroclinic orbits for equations, where the forced symmetry-breaking is not too strong ($\varepsilon > 0$ small). However, in all these examples we found no heteroclinic cycle connecting only the equilibria which were forced by our symmetry (these are for $L = \mathbb{T}$ only the D_2 fixed-points).

The reason for this is simply, that even our perturbed equation still possesses variational structure. At this point however, a more convenient way to understand that problem is to look at (6.3). In case that $\bar{p} \in \bar{\mathcal{R}}^{\mathbb{T}}$, necessarily

$$(\bar{p}, \omega_l(0)^k)_{L^2(S^2)} = (\bar{p}, \omega_l(\pi/2)^k)_{L^2(S^2)}$$

holds. Therefore $\frac{d}{d\varphi}(\bar{p}, \omega_l(\varphi)^k)_{L^2(S^2)}$ must vanish somewhere in $(0, \pi/2)$, i.e. $\mathcal{F}^{(\bar{p},k)}_{\Upsilon_l^{\omega},\mathbb{T}}$ will have an additional zero. We conclude that, in order to see heteroclinic cycles, we have to look at perturbations of a different structure.

7 Heteroclinic Cycles

We now want to consider perturbations of non-variational structure. After all that preliminary work our mission to find heteroclinic cycles will now easily be accomplished.

7.1 Perturbation of the Diffusion Term

In case that p is a $\mathbb{T}-$invariant polynomial on $\mathbb{R}^3$, we obtain that

$$\begin{array}{rcl} B(\varepsilon) : D \subset L^2(\mathbb{R}^3) & \to & L^2(\mathbb{R}^3) \\ u & \mapsto & \operatorname{div}((1+\varepsilon p)\nabla u) \end{array} \tag{7.1}$$

is $\mathbb{T}-$equivariant. Expanding $B(\varepsilon)$, we find that the solutions of $B(\varepsilon)u + f(u) = 0$ solve

$$(1+\varepsilon p)\Delta u + \varepsilon < \nabla p, \nabla u > + f(u) = 0,$$

(where $< \cdot, \cdot >$ is the scalar product in $\mathbb{R}^3$) or,

$$\Delta u + \varepsilon < \nabla p, \nabla u > + (1-\varepsilon p) f(u) = o(\varepsilon).$$

The perturbation of the reaction term is not very helpful for finding heteroclinic cycles, as we saw in the last section. We therefore consider $u \mapsto < \nabla p, \nabla u >$, or, more general

$$\begin{array}{rcl} D \subset L^2(\mathbb{R}^3) & \to & L^2(\mathbb{R}^3) \\ u & \mapsto & q \cdot < \nabla p, \nabla u > \end{array}$$

with q and $p \in \mathcal{R}^{\mathbb{T}}$ as a $\mathbb{T}-$equivariant mapping. This kind of perturbation is achieved (in part) by multiplying (7.1) with $(1+\varepsilon q)$. Obviously, any function $\bar{u} \in L^2(S^2)$ might be extended (at least to an annulus) by $u(x,y,z) := \bar{u}(x/r, y/r, z/r)$, $r = |(x,y,z)|$. Therefore, the restriction

$$\begin{array}{rcl} h : D \subset L^2(S^2) & \rightarrow & L^2(S^2) \\ \bar{u} & \mapsto & \bar{q} \cdot < \nabla \bar{p}, \nabla \bar{u} > \end{array} \tag{7.2}$$

with $\bar{q}$ and $\bar{p} \in \bar{\mathcal{R}}^{\mathbb{T}}$ is a $\mathbb{T}-$equivariant mapping (of $L^2(S^2)$) as well. In the sequel we consider such mappings as perturbations for (1.1) (cf. also (1.8)). For convenience, we note that the gradient of a restriction $\bar{u} := u_{|S^2}$ of a smooth $L^2(\mathbb{R}^3)$ function can be obtained by projecting the gradient of u to the tangent space of the sphere

$$\nabla \bar{u} \begin{pmatrix} x \\ y \\ z \end{pmatrix} = \nabla u \begin{pmatrix} x \\ y \\ z \end{pmatrix} - < \nabla u \begin{pmatrix} x \\ y \\ z \end{pmatrix}, \begin{pmatrix} x \\ y \\ z \end{pmatrix} > \cdot \begin{pmatrix} x \\ y \\ z \end{pmatrix}, \quad \begin{pmatrix} x \\ y \\ z \end{pmatrix} \in S^2.$$

This is the kind of gradient, we have to plug into h, because our functions are usually obtained from restrictions of functions defined on $\mathbb{R}^3$. On the connection $\Upsilon_l^{\omega,\mathbb{T}}$ (cf. (5.8)) we find for the flow (direction) (2.22)

$$\mathcal{F}^h_{\Upsilon_l^{\omega,\mathbb{T}}}(\varphi) := \int_{S^2} \frac{d}{d\varphi}\omega_l(\varphi) \cdot h(\omega_l(\varphi)) dS, \quad \varphi \in [0, \pi/2],$$

considered in Theorem 2.2:

Theorem 7.1 *Using $\bar{q} = \bar{\tau}_6$ and $\bar{p} = \bar{\rho}_4$ for h defined in (7.2) the flow (direction) for the $l = 2$ representation is*

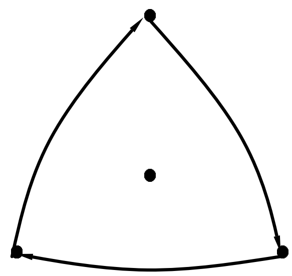
$$\mathcal{F}^h_{\Upsilon_2^{\omega,\mathbb{T}}}(\varphi) = \frac{1024}{5005}\pi \sin(\varphi)\cos(\varphi), \quad \varphi \in [0, \pi/2].$$

Thus, under the assumptions of Theorem 2.2 ($L = \mathbb{T}$ and $H = \mathbf{O}(2) \oplus \mathbb{Z}_2^c$) for the $l = 2$ representation of $\ker A(\lambda_0)$, we find for the semilinear parabolic equation (1.8) with perturbation (7.2), $\bar{q} = \bar{\tau}_6$ and $\bar{p} = \bar{\rho}_4$, a heteroclinic cycle for the perturbed flow. This basic flow $\mathcal{G}_1^{\mathbb{T}}(\varphi) := \sin(\varphi)\cos(\varphi)$ is illustrated in the next figure.

Proof. Simple computation as already used in Section 6 or use Maple. For details confer Section 9. $\square$

Some remarks are in order.

Figure 11: $\mathcal{G}_1^{\mathbb{T}}$



Remark 7.2 *The flow $\mathcal{G}_1^{\mathbb{T}}$ is of a quite stable structure (against $\mathbb{T}-$equivariant perturbations, i.e. perturbations of the form $\varepsilon h(u)+\varepsilon^2\tilde{h}(u)$ with any other $\mathbb{T}-$equivariant mapping $\tilde{h}$, yields, for $\varepsilon > 0$ small enough, again the heteroclinic cycle.*

Remark 7.3 *Other pairs of polynomials $(\bar{q};\bar{p})$, which give (up to a multiple) the $\mathcal{G}_1^{\mathbb{T}}$ flow are e.g. $(\bar{\rho}_4;\bar{\tau}_6)$, $(\bar{\rho}_6;\bar{\tau}_6)$, $(\bar{\tau}_6;\bar{\rho}_6)$, $(\bar{\tau}_6;\bar{\rho}_4^2)$, $(\bar{\rho}_4^2;\bar{\tau}_6)$, $(\bar{\rho}_4;\bar{\rho}_4\bar{\tau}_6)$, $(\bar{\rho}_4\bar{\tau}_6;\bar{\rho}_4)$, $(\bar{\rho}_4^2\bar{\rho}_6;\bar{w}_{14}^{\bar{\mathcal{R}}})$, and $(\bar{w}_{14}^{\bar{\mathcal{R}}};\bar{\rho}_4^2\bar{\rho}_6)$ for $l = 2$. Flows which still give heteroclinic cycles, but which are not exactly the $\mathcal{G}_1^{\mathbb{T}}$ flow are achieved for instance by $(\bar{\tau}_6;\bar{\rho}_4)$, $(\bar{\tau}_6;\bar{\rho}_6)$, and $(\bar{\tau}_6;\bar{\rho}_4^2)$ in case $l = 3$ (i.e. $H = \mathbf{O}(2)^-$).*

Remark 7.4 *Despite some computational effort and using the knowledge of the space $W^{\bar{\mathcal{R}}}$, we have not been able to find heteroclinic cycles for $h(\bar{u}) = \bar{p}^m \cdot \nabla\bar{p}\nabla\bar{u}\cdot\bar{u}^k$ with $m,k \in \mathbb{N}_0$. However, we still find all heteroclinic orbits of Section 6.*

7.2 Perturbations using $\mathbb{T}-$Equivariant Polynomial Mappings

An obvious generalization of the perturbation (7.2) is

$$\begin{array}{rcl} h : D \subset L^2(S^2) & \to & L^2(S^2) \\ \bar{u} & \mapsto & <\bar{\epsilon},\nabla\bar{u}> \cdot\bar{u}^k \end{array} \tag{7.3}$$

with some $\bar{\epsilon} \in \bar{\mathcal{M}}^{\mathbb{T}}$. Since $\bar{\epsilon} : S^2 \to \mathbb{R}^3$ is $\mathbb{T}-$equivariant, it follows easily that h is $\mathbb{T}-$equivariant as well. Section 4.2 was devoted to the question which elements are precisely $\mathbb{T}-$equivariant. These are the elements in $W^{\bar{\mathcal{M}}}$ (cf.Theorem 4.19), two of which are $\bar{w}_3^{\bar{\mathcal{M}}}$ and $\bar{w}_7^{\bar{\mathcal{M}}}$.

Using $\bar{\epsilon} := \bar{w}_3^{\bar{\mathcal{M}}}$ and $k = 0$ for h defined in (7.3) the flow for the $l = 2$ representation is

$$\mathcal{F}^h_{\Upsilon_2^\omega,\mathbb{T}}(\varphi) = \frac{96}{35}\pi\cos(\varphi)\sin(\varphi), \quad \varphi \in [0,\frac{\pi}{2}].$$

This is again a $\mathcal{G}_1^{\mathbb{T}}$ flow. Another tetrahedral flow can be observed with $\bar{\epsilon} := \bar{w}_7^{\bar{\mathcal{M}}}$ and $k = 2$ for h defined in (7.3). The flow for the $l = 2$ representation is

$$\mathcal{F}^h_{\Upsilon_2^\omega,\mathbb{T}}(\varphi) = \frac{41472}{55055}\pi\cos(\varphi)\sin(\varphi) - \frac{13824}{5005}\pi\cos^3(\varphi)\sin(\varphi) + \frac{13824}{5005}\pi\cos^5(\varphi)\sin(\varphi). \quad (7.4)$$

This gives a combination of $\mathcal{G}_1^{\mathbb{T}}$ with the basic flow $\mathcal{G}_2^{\mathbb{T}}(\varphi) := \sin^3(\varphi)\cos^3(\varphi)$. Although Theorem 2.2 is not directly applicable to $\mathcal{G}_2^{\mathbb{T}}$, it is applicable to the flow in (7.4) giving qualitatively again the picture in Figure 11.

Remark 7.5 *Other pairs $(l;k)$, which give together with $\bar{w}_3^{\bar{\mathcal{M}}}$ the $\mathcal{G}_1^{\mathbb{T}}$ flow (up to multiples) are e.g.* $(3;0)$, $(4;0)$, $(2;1)$, $(2;2)$, $(2;3)$, *and* $(4;1)$. *The $\mathcal{G}_2^{\mathbb{T}}$ flow combined with $\mathcal{G}_1^{\mathbb{T}}$ as in (7.4) can also be observed for $\bar{w}_7^{\bar{\mathcal{M}}}$ with the following pairs $(l;k)$:* $(2;3)$, $(2;4)$, $(3;2)$, $(4;0)$, $(4;1)$, *and* $(4;2)$. *Of course these lists are by no means complete.*

8 Applications to Reaction Diffusion Systems

Here we want to address the question of applying the previous results to systems. As an example we discuss the equations of the brusselator on the 2–sphere S^2_ρ of radius ρ. We consider these equations to be a test case for more interesting equations. Our equations have the following form

$$\begin{aligned} \frac{\partial U}{\partial t} &= D_1\Delta U + U^2V - (B+1)U + A \\ \frac{\partial V}{\partial t} &= D_2\Delta V - U^2V + BU, \end{aligned} \quad (8.1)$$

where D_1, D_2 are positive and $A, B \in \mathbb{R}$ (compare Golubitsky and Schaeffer [4], Chapter VII §5). We find easily a family of spatially constant equilibria, namely

$$U = A \text{ and } V = B/A. \quad (8.2)$$

Usually one considers B to be a control parameter, while D_1, D_2, A and ρ are fixed. The stability analysis for this family of equilibria is the same as the stability analysis for the brusselator discussed in [4]. Therefore we just present the results. If one considers spatially constant perturbations only, one has to consider the system of ODE's

$$\begin{aligned} \frac{\partial U}{\partial t} &= U^2V - (B+1)U + A \\ \frac{\partial V}{\partial t} &= -U^2V + BU. \end{aligned}$$

The family discussed before is stable if $B < 1 + A^2$. At $B = 1 + A^2$ we have a Hopf bifurcation and a family of spatially constant periodic solutions occurs. If we are interested in stable non-spatially constant solutions we have to consider the full system of PDE's. The system under consideration is obviously $\mathbf{O}(3)$–equivariant. If we look for points where the family (8.2) looses the stability it is natural to ask which representation of $\mathbf{O}(3)$ occurs on the eigenspace corresponding to purely imaginary eigenvalues.

By changing the parameters one can also find other interesting bifurcations. In fact we show

Theorem 8.1 *For each $\ell \in \mathbb{N}$ there exist diffusion constants D_1, D_2 and parameters A, ρ and a critical number B_ℓ such that for $B < B_\ell$ the trivial solution (8.2) is linearly stable, and unstable for $B > B_\ell$. Moreover, for $B = B_\ell$ the kernel of the linearization at the trivial solution is the absolutely irreducible representation of $\mathbf{O}(3)$ of dimension $2\ell + 1$.*

Proof. The proof of this theorem proceeds along the lines of the proof in [4]. Write $U = A + u,\ V = \frac{B}{A} + v$, then the system (8.1) takes the form

$$\begin{aligned} \frac{\partial u}{\partial t} &= D_1 \Delta u + (B-1)u + A^2 v + f(u,v) \\ \frac{\partial v}{\partial t} &= D_2 \Delta v - Bu - A^2 v - f(u,v), \end{aligned} \tag{8.3}$$

where f is given by $f(u,v) = \frac{B}{A}u^2 + 2Auv + u^2 v$. Let $Y_m^\ell,\ m = -\ell, \ldots, \ell$ be the spherical harmonics of order ℓ. The Laplace operator applied to Y_m^ℓ considered on the sphere of radius ρ gives

$$\Delta Y_m^\ell = \frac{\ell(\ell+1)}{\rho^2} Y_m^\ell. \tag{8.4}$$

Therefore the linearization of (8.3) leads to

$$\begin{aligned} \frac{\partial u}{\partial t} &= D_1 \Delta u + (B-1)u + A^2 v \\ \frac{\partial v}{\partial t} &= D_2 \Delta v - Bu - A^2 v, \end{aligned} \tag{8.5}$$

and the eigenfunctions of this system have the form

$$Y = Y_m^\ell \begin{pmatrix} u_0 \\ v_0 \end{pmatrix}. \tag{8.6}$$

For Y to be an eigenvector the vector $\begin{pmatrix} u_0 \\ v_0 \end{pmatrix}$ has to satisfy the condition

$$\begin{pmatrix} \mu D_1 + (B-1) & A^2 \\ -B & \mu D_2 - A^2 \end{pmatrix} \begin{pmatrix} u_0 \\ v_0 \end{pmatrix} = \lambda \begin{pmatrix} u_0 \\ v_0 \end{pmatrix} \tag{8.7}$$

where $\mu(\ell) = \frac{\ell(\ell+1)}{\rho^2}$. Looking for steady state bifurcations means that we set

$$\det \begin{bmatrix} \mu D_1 + (B-1) & A^2 \\ -B & \mu D_2 - A^2 \end{bmatrix} = 0. \tag{8.8}$$

In order to prove the theorem we have to show that for given $l_0 \in \mathbb{N}$ the parameters A, D_1, D_2, and ρ can be arranged such that there exists a number $B_{\ell_0} < 1 + A^2$ such that for $B < B_{\ell_0}$ the given branch (8.2) is stable, for $B = B_{\ell_0}$ there exists some solution to (8.8) with $\mu(\ell) = \mu(\ell_0)$ and for all other ℓ the determinant is positive. Moreover the kernel of (8.7) is one-dimensional. It is just a matter of some computations to verify these claims. □

Choosing the parameters as $D_1 = 1$, $D_2 = 4$, $A = 3$, $\rho = 2$, and $B = B_2 = 77/8$ we get the 5–dimensional irreducible representation of $\mathbf{O}(3)$ as the one through which the trivial solution looses its stability.

We consider symmetry-breaking perturbations of the following type

$$\varepsilon \begin{pmatrix} h_1(B, u, \nabla u, x) \\ h_2(B, u, \nabla u, x) \end{pmatrix} \tag{8.9}$$

with $h_1(B, u, \nabla u, x) =< \overline{\epsilon_1}, \nabla u >$ and $h_2(B, u, \nabla u, x) =< \overline{\epsilon_2}, \nabla u >$, where $\overline{\epsilon}_{1,2} \in \overline{\mathcal{M}}^{\mathbb{T}}$. In order to apply the methods developed in this paper we calculate the arcs Υ within function space $L^2(S^2)$. In order to get the drift along these arcs we have to compute the scalar product between the tangent vectors and the perturbation terms, as we have seen in Section 2 and 7. The computations are the same as in the previous cases, therefore we just state the results.

Theorem 8.2 *There exist perturbations of the form (8.9) of degree 3, and an $\varepsilon_0 > 0$, such that for each perturbation with $\varepsilon < \varepsilon_0$ there exist heteroclinic cycles, as described before.*

9 Appendix: Computation of Flows Using Maple

To calculate the flow formula (2.22) we have to find a way to integrate efficiently over the sphere S^2. We will outline, how the symbolic computation program *Maple* (actually we use *Maple V,* Release 2) can be used to integrate polynomials $p = p(x, y, z)$ over S^2. Writing

$$p(x, y, z) = \sum_{i,j,m} \alpha_{i,j,m} x^i y^j z^m \text{ and } \vartheta(i, j, m) := \int_{S^2} x^i y^j z^m dS,$$

we find

$$\int_{S^2} p(x,y,z)dS = \sum_{i,j,m} \alpha_{i,j,m}\vartheta(i,j,m).$$

Therefore the knowledge of the numbers $\vartheta(i,j,m)$ is crucial for our problem. As already remarked in (6.6) and (6.7), we have for any permutation σ of (i,j,m)

$$\vartheta(i,j,m) = \vartheta(\sigma(i),\sigma(j),\sigma(m)), \tag{9.1}$$

and

$$\vartheta(i,j,m) = 0 \text{ for } i,j,m \in \mathbb{N}_0 \text{ and } i,j \text{ or } m \text{ odd}.$$

Hence only $\vartheta(i,j,m)$ for i,j and m even is of interest. The recursion formula

$$\vartheta(i,j,m+2) = \frac{m+1}{i+j+m+3}\vartheta(i,j,m), \quad i,j,m \in \mathbb{N}_0$$

is not hard to see. Using (9.1) we get similar expressions for increasing i and j, whence all needed values of ϑ can be calculated easily using $\vartheta(0,0,0) = vol(S^2) = 4\pi$. Provided for even $i \leq j \leq m$ the values of integ$([i,j,m]) := \vartheta(i,j,m)$ are known, the following maple procedure will calculate $\int_{S^2} p\ dS$.

```
polyint:= proc(p)
  local q,value,dx,dy,dz, set,s,t;
   # the values for integ(i,j,m) must be known
  value:= 0;
  simplify(p); expand(");
  collect(",[x,y,z],distributed); q:=combine(");
  while (q<>0)
    do
      s:=lcoeff(q,[x,y,z],'t'); q-s*t;  # extracts one coefficient x^i y^j z^m of q
      q:=simplify(");
      if s*t=0 then  q:= combine(");    # usually not necessary
      else
        dx:= degree(t,x);
        if type(dx,even) then       # only x^i y^j z^m with i,j,m even needed
          dy:= degree(t,y);
          if type(dy,even)  then
            dz:= degree(t,z);
            if type(dz,even) then
              set := [dx,dy,dz];    # this coefficient yields a nontrivial integral
              set:=sort(set);
              value:= value+ s* integ(set);
```

```
            fi:  # {dz}
          fi:  # {dy}
        fi:  # {dx}
      fi: # {else}
    od:
  value:=simplify(value);
  RETURN(value);
end;
```

To obtain the flows of Section 6 e.g., we use:

```
flow:=proc(p,k)                 #Input p=polynomial,  k=integer
  local wdiff, prod;            #w=w_l(phi) must be known
  wdiff:=diff(w,phi);
  prod := wdiff*p*w^k;
  subs(cos(phi)=ccc,"); subs(sin(phi)=sss,");
  prod:="; value:= polyint(prod);
  subs(ccc=cos(phi),"); subs(sss=sin(phi),");
  value:=simplify("); RETURN(value);
end;
```

Here a parametrization of a connection, for instance w $= \omega_l(\varphi)$ (see Section 5), must be known. The sub- and resubstitution of $\sin(\varphi)$ and $\cos(\varphi)$ is not really necessary, but it speeds things up enormously.

Contents

List of Figures

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